1 Limits at ∞

Rational functions sometimes have horizontal asymptotes. We find and describe these asymptotes using limits at ∞ (and limits at −∞). The standard technique is to divide the numerator and denominator by the largest power of \( x \) in the denominator.

The following function has a horizontal asymptote at \( y = 2 \).

\[
\lim_{x \to \infty} \frac{x - 4x^5 + 1}{-2x^5 - 7} = \lim_{x \to \infty} \left( \frac{x - 4x^5 + 1}{x^5} \right) \frac{1}{\frac{-2 - \frac{7}{x^5}}{x^5}} = \frac{0 - 4 + 0}{-2 - 0} = 2
\]

Note that \( \lim_{x \to -\infty} \frac{x - 4x^5 + 1}{-2x^5 - 7} = 2 \) also. To see this, you use exactly the same technique.

\[
\lim_{x \to -\infty} \frac{x - 4x^5 + 1}{-2x^5 - 7} = \lim_{x \to -\infty} \left( \frac{x - 4x^5 + 1}{x^5} \right) \frac{1}{\frac{-2 - \frac{7}{x^5}}{x^5}} = \frac{0 - 4 + 0}{-2 - 0} = 2
\]

The next function has a horizontal asymptote at \( y = 0 \).

\[
\lim_{x \to \infty} \frac{3x + 1}{x^2 - 1} = \lim_{x \to \infty} \left( \frac{3x + 1}{x^2} \right) \frac{1}{1 - \frac{1}{x^2}} = \frac{0 + 0}{1 - 0} = 0
\]

Again, we get the same limit as \( x \to -\infty \).

\[
\lim_{x \to -\infty} \frac{3x + 1}{x^2 - 1} = \lim_{x \to -\infty} \left( \frac{3x + 1}{x^2} \right) \frac{1}{1 - \frac{1}{x^2}} = \frac{0 + 0}{1 - 0} = 0
\]

If you don’t want to have to write so much twice, then use following notation most concise.

\[
\lim_{x \to \pm\infty} \frac{3x + 1}{x^2 - 1} = \lim_{x \to \pm\infty} \left( \frac{3x + 1}{x^2} \right) \frac{1}{1 - \frac{1}{x^2}} = \frac{0 + 0}{1 - 0} = 0
\]

Some non-rational functions have “one-sided” horizontal asymptotes. For example, \( \lim_{x \to \infty} \frac{x|x|}{x^2 + 5} = 1 \) and \( \lim_{x \to -\infty} \frac{x|x|}{x^2 + 5} = -1 \). (Exercise: why?)

Here’s a rational function without any horizontal asymptote.

\[
\lim_{x \to \infty} \frac{3x^5 + 5x - 8}{-6x^2 - 1} = \lim_{x \to \infty} \left( \frac{3x^5 + 5x - 8}{x^2} \right) \frac{1}{\frac{-6 - \frac{1}{x^2}}{x^2}} = \frac{3(+big)^3 + 0 - 0}{-6 - 0} = \frac{+big}{-6} = -\infty
\]

Note that the limit as \( x \to -\infty \) is slightly different.

\[
\lim_{x \to -\infty} \frac{3x^5 + 5x - 8}{-6x^2 - 1} = \lim_{x \to -\infty} \left( \frac{3x^5 + 5x - 8}{x^2} \right) \frac{1}{\frac{-6 - \frac{1}{x^2}}{x^2}} = \frac{3(-big)^3 + 0 - 0}{-6 - 0} = \frac{-big}{-6} = \infty
\]
2 Vertical asymptotes

Division by zero produce finite limits when the zero divisors are cancelled by factors of the numerator.

\[
\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \lim_{x \to 3} \frac{(x + 1)(x - 3)}{(x - 1)(x - 3)} = \lim_{x \to 3} \frac{x + 1}{x - 1} = \frac{3 + 1}{3 - 1} = 2
\]

When cancellation fails, there is a vertical asymptote and the limit fails to exist. The interesting question is how the limit fails to exist. The general pattern is that if \( f(x) \to L \neq 0 \) and \( g(x) \to 0 \) as \( x \to a \), then \( f(x)/g(x) \) gets very large as \( x \) approaches \( a \). But does \( f(x)/g(x) \) get large and positive or large and negative as \( x \to a \)? Both? Does the answer change if \( x \) only approaches \( a \) from the left or the right?

The following function gets large and positive as \( x \) approaches 3 from the right, but gets large and negative as \( x \) approaches 3 from the left.

\[
\lim_{x \to 3^+} \frac{x^2 - 2x - 3}{x^2 - 6x + 9} = \lim_{x \to 3^+} \frac{(x + 1)(x - 3)}{(x - 3)^2} = \lim_{x \to 3^+} \frac{x + 1}{(x - 3)} = \frac{3 + \text{small} + 1}{3 + \text{small} - 3} = \frac{4}{\text{small}} = \infty
\]

\[
\lim_{x \to 3^-} \frac{x^2 - 2x - 3}{x^2 - 6x + 9} = \lim_{x \to 3^-} \frac{(x + 1)(x - 3)}{(x - 3)^2} = \lim_{x \to 3^-} \frac{x + 1}{x - 3} = \frac{3 - \text{small} + 1}{3 - \text{small} - 3} = \frac{4}{-\text{small}} = -\infty
\]

The following function gets large and negative as \( x \) approaches 3 from both directions.

\[
\lim_{x \to 3} \frac{x^2 + 1}{-x^2 + 6x - 9} = \lim_{x \to 3} \frac{x^2 + 1}{-(x - 3)^2} = \frac{(3 \pm \text{small})^2 + 1}{-(3 \pm \text{small})^2} = \frac{10}{-\text{small}} = -\infty
\]

An alternative way to compute the above limits is to make sign tables.

\[
\begin{array}{cccc}
-1 & 3 \\
+ & + + + \\
- & - - 0 + \\
+ & 0 - * + \\
\end{array}
\]

Thus, \( \lim_{x \to 3^+} \frac{x + 1}{x - 3} = \infty \) because for \( x \) close to 3 from the right, \( \frac{x + 3}{x - 3} \) is positive. By similar reasoning, \( \lim_{x \to 3^-} \frac{x + 1}{x - 3} = -\infty \).