[University Physics I (2325)] CIRCULAR MOTION: A POLAR COORDINATE APPROACH

DAVID MILOVICH

Algebraically speaking, a **circle** in the plane is the set of solutions (x, y) to an equation of the form

$$(x-a)^2 + (y-b)^2 = r^2$$

where a, b, and r are constants and $r > 0$. (Note that x, y, a, b, and r all have dimensions $[L]$, meaning they are all lengths.) Using trigonometry, especially the identity $\cos^2\theta + \sin^2\theta = 1$, one can show that the solution set is exactly the set of points (x, y) that can be written as

$$(x, y) = (a + r\cos\theta, b + r\sin\theta)$$

for some real number θ.

Geometrically, (a, b) is the **center** of our circle, and r is the **radius**. θ is the **angular position**. More precisely, let $C = (a, b)$ be the center of our circle, let P be the rightmost point $(a + r, b)$ on our circle, and let Q be the point $(a + r\cos\theta, b + r\sin\theta)$. The angle $\angle PCQ$ corresponds to θ in the following sense. If θ is positve and we rotate the line segment $\overline{CP}$ counterclockwise by θ radians about the center C, then we obtain the line segment $\overline{CQ}$. If θ is negative and we rotate the line segment $\overline{CP}$ *clockwise* by $-\theta$ radians about the center C, then we obtain the line segment $\overline{CQ}$. If $\theta = 0$, then $P = Q$, so $\overline{CP} = \overline{CQ}$. Therefore, if θ is less than 2π in magnitude, then $\angle PCQ = \pm\theta$. On the other hand, if θ can be greater than 2π in magnitude, then we have to subtract multiples of 2π from $\pm\theta$ in to obtain $\angle PCQ$. This is because, for example, rotating something 27° produces the same orientation at rotating that something 387°.

Now suppose that a particle is moving along our circle. Let $(x(t), y(t))$ be its position at time t. There then exists a function $\theta(t)$ such that

$$(x(t), y(t)) = (a + r\cos\theta(t), b + r\sin\theta(t)).$$

Thinking of $(x(t), y(t))$ as position relative to $(0, 0)$, let us use vector notation: $(x(t), y(t)) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$. Differentiating $x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$ with respect to t, we find that the **velocity** is

$$\vec{\boldsymbol{v}} = -r\frac{d\theta}{dt}(\sin\theta(t))\hat{\boldsymbol{\imath}} + r\frac{d\theta}{dt}(\cos\theta(t))\hat{\boldsymbol{\jmath}}.$$

Therefore, the **speed** is

$$v = \sqrt{\left(-r\frac{d\theta}{dt}\sin\theta(t)\right)^2 + \left(r\frac{d\theta}{dt}\cos\theta(t)\right)^2} = \sqrt{\left(r\frac{d\theta}{dt}\right)^2\left(\sin^2\theta(t) + \cos^2\theta(t)\right)} = \sqrt{\left(r\frac{d\theta}{dt}\right)^2} = r\left|\frac{d\theta}{dt}\right|.$$

r is a constant, but $\frac{d\theta}{dt}$ is general not a constant. Therefore, speed is generally not constant in circular motion. However, it is better to start with a simple model before generalizing to a more complictation model. Therefore, let us restrict our attention to **uniform circular motion**, which is circular motion with constant speed. The only way the speed $r\left|\frac{d\theta}{dt}\right|$ can be constant is if $\frac{d\theta}{dt}$ is constant.

Following tradition, let us define $\omega = \frac{d\theta}{dt}$. (There is also a term for $\frac{d\theta}{dt}$: angular velocity.) This abbreviation allows us to more compactly express the speed as $r|\omega|$ and the velocity as

$-r\omega(\sin\theta(t))\hat{\boldsymbol{\imath}} + r\omega(\cos\theta(t))\hat{\boldsymbol{\jmath}}$. Differentiating velocity with respect to time, we find that the acceleration is

$$\vec{\boldsymbol{a}} = -r\omega\frac{d\theta}{dt}(\cos\theta(t))\hat{\boldsymbol{\imath}} - r\omega\frac{d\theta}{dt}(\sin\theta(t))\hat{\boldsymbol{\jmath}} = -r\omega^2(\cos\theta(t))\hat{\boldsymbol{\imath}} - r\omega^2(\sin\theta(t))\hat{\boldsymbol{\jmath}}.$$

Therefore, the magnitude of accelaration is

$$a = \sqrt{(-r\omega^2\cos\theta(t))^2 + (-r\omega^2\sin\theta(t))^2} = \sqrt{(-r\omega^2)^2\left(\cos^2\theta(t) + \sin^2\theta(t)\right)} = \sqrt{(-r\omega^2)^2} = r\omega^2.$$

Since the speed is $v = r|\omega|$, we have $\omega = \pm v/r$. Therefore,

$$a = r\omega^2 = r\left(\pm\frac{v}{r}\right)^2 = \frac{v^2}{r}.$$

Moreover, $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{v}}$ are **perpendicular**. Physically, this must be true: if acceleration is not perpendicular to velocity, then the speed either increases or decreases, in contradiction with assumption of *uniform* circular motion, *i.e.*, constant speed. Algebraically, the vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ are perpendicular if their slopes s_v and s_a are reciprocal opposites, *i.e.*, $s_v = -1/s_a$. The slope s_a of the acceleration vector is given by dividing its y-component by its x-component:

$$s_a = \frac{-r\omega^2\sin\theta(t)}{-r\omega^2\cos\theta(t)} = \frac{\sin\theta(t)}{\cos\theta(t)}.$$

Likewise,

$$s_v = \frac{r\omega\cos\theta(t)}{-r\omega\sin\theta(t)} = -\frac{\cos\theta(t)}{\sin\theta(t)} = -\frac{1}{s_a}.$$

Geometrically, $\vec{\boldsymbol{a}}$ is an arrow that we can slide such that it points from $(x(t), y(t))$ towards the center (a, b) of the circle. (One can check this algebraically: $(a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}) - (x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}) = -r\cos\theta(t)\hat{\boldsymbol{\imath}} - r\sin\theta(t)\hat{\boldsymbol{\jmath}} = \frac{1}{\omega^2}\vec{\boldsymbol{a}}$.) in other words, $\vec{\boldsymbol{a}}$ points inward along a radius. On the other hand, $\vec{\boldsymbol{v}}$ is an arrow that, if we slide it to start at $(x(t), y(t))$, must point tangent to the circle. (That $\vec{\boldsymbol{v}}$ points tangent to the circle at position $(x(t), y(t))$ can proven mathematically from the definition of derivative, or can be argued physically by considering how nontangential velocity would cause our particle to go either inside or outside our circle.) Finally, as has been known since (at least) the time of the ancient Greeks, whenever a tangent to a circle and a radius of that circle intersect at a point on that circle, they intersect at a right angle. Thus, $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{v}}$ are perpendicular.