Question. Imagine discs sliding on a flat sheet of ice. Disc A, with mass \( m_A = 0.35 \text{kg} \) and initial velocity \( \vec{v}_A = 0.75 \hat{j} (m/s) \), hits disc B, which has mass \( m_B = 0.95 \text{kg} \) and initial speed \( v_B = 0.00 \text{m/s} \). After the collision, disc B has final velocity \( \vec{v}_B \) directed \( \theta = 42^\circ \) clockwise from \( \hat{j} \).

For simplicity, assume that the collision is elastic and that friction (and all other external forces) and spin are negligible. What are the final velocities \( \vec{v}_A^* \) and \( \vec{v}_B^* \)?

Answer. By definition of elastic collision, the total kinetic energy is conserved:

\[
\frac{1}{2} m_A (v_A^*)^2 + \frac{1}{2} m_B (v_B^*)^2 = K_A = K_B = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2.
\]

Since all external forces are negligible, the total momentum is conserved:

\[
m_A \vec{v}_A + m_B \vec{v}_B = \vec{p}_A + \vec{p}_B = \vec{p}_A + \vec{p}_B = m_A \vec{v}_A + m_B \vec{v}_B.
\]

Break up this vector equation into \( x \) and \( y \) components.

\[
\begin{align*}
m_A v_{Ax}^* + m_B v_{Bx}^* &= p_A^x + p_B^x = p_A^x + p_B^x = m_A v_{Ax} + m_B v_{Bx} \\
m_A v_{Ay}^* + m_B v_{By}^* &= p_A^y + p_B^y = p_A^y + p_B^y = m_A v_{Ay} + m_B v_{By}
\end{align*}
\]

Thus, we have three equations and apparently six unknowns, \( v_{Ax}^*, v_{Ay}^*, v_{Bx}^*, v_{By}^* \). However, we can use \( \theta \) and a little geometry get the remaining equations we need. By elementary trigonometry, \( v_{Bx}^* = v_B^* \sin \theta \) and \( v_{By}^* = v_B^* \cos \theta \). Also, note that \( v_{Ax} = v_A, v_{Ay} = 0, \) and \( v_{Bx} = v_{By} = v_B = 0 \).

Our three equations now simplify:

\[
\begin{align*}
\frac{1}{2} m_A (v_A^*)^2 + \frac{1}{2} m_B (v_B^*)^2 &= \frac{1}{2} m_A v_A^2 \\
m_A v_{Ax}^* + m_B v_{Bx}^* \sin \theta &= m_A v_A \\
m_A v_{Ay}^* + m_B v_{By}^* \cos \theta &= 0
\end{align*}
\]

Solve the last two equations for \( v_{Ax}^* \) and \( v_{Ay}^* \):

\[
\begin{align*}
v_{Ax}^* &= v_A - m_A^{-1} m_B v_B^* \sin \theta \\
v_{Ay}^* &= -m_A^{-1} m_B v_B^* \cos \theta
\end{align*}
\]

Apply the Pythagorean Theorem to \( \vec{v}_A^* \):

\[
(v_A^*)^2 = (v_{Ax}^*)^2 + (v_{Ay}^*)^2 = (v_A - m_A^{-1} m_B v_B^* \sin \theta)^2 + (-m_A^{-1} m_B v_B^* \cos \theta)^2
\]

\[
= v_A^2 - 2 v_A m_A^{-1} m_B v_B^* \sin \theta + m_A^{-2} m_B^2 (v_B^*)^2 \sin^2 \theta + m_A^{-2} m_B^2 (v_B^*)^2 \cos^2 \theta
\]

\[
= v_A^2 - 2 v_A m_A^{-1} m_B v_B^* \sin \theta + m_A^2 m_B^2 (v_B^*)^2
\]
Substitute this formula for \((v_A^*)^2\) into our simplified equation for the conservation of kinetic energy and solve for \(v_B^*\):

\[
\begin{align*}
\frac{1}{2}m_Av_A^2 &= \frac{1}{2}m_A(v_A^*)^2 + \frac{1}{2}m_B(v_B^*)^2 \\
\frac{1}{2}m_Av_A^2 &= \frac{1}{2}m_A(v_A^* - 2v_Am_A^{-1}m_Bv_B^*)^2 + m_A^{-2}m_B^2(v_B^*)^2 + \frac{1}{2}m_B(v_B^*)^2 \\
m_Av_A^2 &= m_A(v_A^* - 2v_Am_A^{-1}m_Bv_B^*)^2 + m_A^{-2}m_B^2(v_B^*)^2 + m_B(v_B^*)^2 \\
m_Av_A^2 &= m_A^2v_A^2 - 2v_Am_Bv_B^*\sin \theta + m_A^{-1}m_B^2(v_B^*)^2 + m_B(v_B^*)^2 \\
0 &= -2v_Am_Bv_B^*\sin \theta + m_A^{-1}m_B^2(v_B^*)^2 + m_B(v_B^*)^2 \\
0 &= v_B(-2v_Am_B\sin \theta + m_A^{-1}m_B^2v_B^* + m_Bv_B^*)
\end{align*}
\]

The product of two quantities is zero exactly when one (or both) of the factors is zero. The first possibility, \(v_B^* = 0\), corresponds to the physical situation in which disc A merely passes by disc B without hitting it. By assumption, disc A actually hits disc B, so \(v_B^* \neq 0\) and the other factor is zero:

\[
\begin{align*}
0 &= -2v_Am_B\sin \theta + m_A^{-1}m_B^2v_B^* + m_Bv_B^* \\
0 &= -2v_Am_B\sin \theta + (m_A^{-1}m_B^2 + m_B)v_B^* \\
v_Am_B\sin \theta &= (m_A^{-1}m_B^2 + m_B)v_B^* \\
v_Am_B\sin \theta &= (m_B^2 + m_Am_B)v_B^* \\
\frac{2v_Am_B\sin \theta}{m_B^2 + m_Am_B} &= v_B^*
\end{align*}
\]

Now plug our solution for \(v_B^*\) into \(v_A^*, v_Ay^*, v_Bx^*, \) and \(v_By^*\). As an optional step, we can simplify using the double-angle formulas \(\cos 2\theta = 1 - 2\sin^2 \theta\) and \(2\sin \theta \cos \theta = \sin 2\theta\).

\[
\begin{align*}
v_A^* &= v_A - m_A^{-1}m_Bv_B^*\sin \theta = v_A - \frac{2v_Am_B\sin^2 \theta}{m_B^2 + m_A} = \frac{v_A(m_B^2 + m_A)}{m_B^2 + m_A} - \frac{2v_Am_B\sin^2 \theta}{m_B^2 + m_A} = \frac{v_A(m_B^2 + m_A \cos 2\theta)}{m_B^2 + m_A} \\
v_A^y &= -m_A^{-1}m_Bv_B^*\cos \theta = -\frac{2v_Am_B\sin \theta \cos \theta}{m_B^2 + m_A} = -\frac{v_Am_B\sin 2\theta}{m_B^2 + m_A} \\
v_Bx^* &= v_B^*\sin \theta = \frac{2v_Am_A\sin^2 \theta}{m_A + m_B} = \frac{v_Am_A(1 - \cos 2\theta)}{m_A + m_B} \\
v_By^* &= v_B^*\cos \theta = \frac{2v_Am_B\sin \theta \cos \theta}{m_B + m_A} = \frac{v_Am_B\sin 2\theta}{m_B + m_A}
\end{align*}
\]

Plugging in our given values for \(m_A, m_B, v_A, \theta\) into our above solutions, we get:

\[
\begin{align*}
\vec{v}_A^* &= v_{Ax}^*\hat{i} + v_{Ay}^*\hat{j} = (0.259213\hat{i} - 0.545075\hat{j})(m/s) \\
\vec{v}_B^* &= v_{Bx}^*\hat{i} + v_{By}^*\hat{j} = (0.180816\hat{i} + 0.200817\hat{j})(m/s)
\end{align*}
\]

Rounding to 2 significant figures, our final answers are \(\vec{v}_A = (0.26\hat{i} - 0.55\hat{j})(m/s)\) and \(\vec{v}_B = (0.18\hat{i} + 0.20\hat{j})(m/s)\).
**Confirmation.** Let us numerically test our answers. The momenta below are in meters per second; the kinetic energies are in Joules.

\[
m_A v_{Ax} + m_B v_{Bx} = (0.35)(0.75) + (0.95)(0) = 0.2625
\]

\[
m_A v_{Ax}^* + m_B v_{Bx}^* = (0.35)(0.259213) + (0.95)(0.180816) = 0.262500
\]

\[
m_A v_{Ay} + m_B v_{By} = (0.35)(0) + (0.95)(0) = 0
\]

\[
m_A v_{Ay}^* + m_B v_{By}^* = (0.35)(-0.545075) + (0.95)(0.200817) = 0.000000
\]

\[
\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = (0.5)(0.35)(0.5626) + (0.5)(0.95)(0) = 0.0984375
\]

\[
\frac{1}{2}m_A (v_A^*)^2 + \frac{1}{2}m_B (v_B^*)^2 = \frac{1}{2}m_A ((v_{Ax}^*)^2 + (v_{Ay}^*)^2) + \frac{1}{2}m_B ((v_{Bx}^*)^2 + (v_{By}^*)^2)
\]

\[
= (0.5)(0.35)(0.0671912 + 0.297106) + (0.5)(0.95)(0.0326946 + 0.0403274)
\]

\[
= 0.0984375
\]