[University Physics I (2325)] ROCKETRY

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Question. An amateur rocket has mass 1.00kg at launch, 0.90kg of which is fuel. The rocket flies straight up, burning its fuel at a constant rate. The burn time in 1.50s; during the burn, the velocity of the exhaust gas relative to the rocket is constantly $35m/s$ straight down. Ignore air resistance and assume that gravitional acceleration is constantly $-9.80m/s^2$. (The latter assumption is justified provided the rocket doesn't get too high.)

(1) What is the velocity of the rocket at the end of the burn?
(2) What is the height of the rocket at the end of the burn?
(3) What is the maximum height attained by the rocket?

Answer. As argued in lecture (and in section 9.10 of the textbook), if the velocity of the rocket is $\vec{\boldsymbol{v}}$ and its mass is M, then

$$M\frac{d\vec{\boldsymbol{v}}}{dt} = \sum \vec{\boldsymbol{F}}_{\text{ext}} + \frac{dM}{dt}\vec{\boldsymbol{u}} = M\vec{\boldsymbol{g}} + \frac{dM}{dt}\vec{\boldsymbol{u}}. \tag{1}$$

where $\sum \vec{\boldsymbol{F}}_{\text{ext}} = M\vec{\boldsymbol{g}}$ is the net external force—just gravity—and $\vec{\boldsymbol{u}}$ is the relative velocity of the exhaust gas.

Let the positive y-direction be up, making $\vec{\boldsymbol{g}} = -9.80\hat{\boldsymbol{\jmath}}(m/s^2)$ and $\vec{\boldsymbol{u}} = -35\hat{\boldsymbol{\jmath}}(m/s)$. We first need to find the change in $\vec{\boldsymbol{v}}$ over the burn time. Multiply equation (1) by $\frac{dt}{M}$:

$$d\vec{\boldsymbol{v}} = \vec{\boldsymbol{g}}\,dt + \vec{\boldsymbol{u}}\frac{dM}{M}. \tag{2}$$

Let time 0 be the start of the burn. Let $\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{0}}$ and $M_1 = 1.00kg$ denote the initial velocity and mass at time $t_1 = 0$. Let $\vec{\boldsymbol{v}}_2$ and $M_2 = 0.10kg$ denote the velocity and mass at time $t_2 = 1.50s$ Integrating (2), we have

$$\begin{aligned}
\int_{\vec{\boldsymbol{v}}_1}^{\vec{\boldsymbol{v}}_2} d\vec{\boldsymbol{v}} &= \vec{\boldsymbol{g}}\int_{t_1}^{t_2} dt + \vec{\boldsymbol{u}}\int_{M_1}^{M_2}\frac{dM}{M} \\
\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1 &= \vec{\boldsymbol{g}}(t_2 - t_1) + \vec{\boldsymbol{u}}\ln\frac{M_2}{M_1} \\
\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{0}} &= -9.80(m/s^2)\hat{\boldsymbol{\jmath}}(1.50s) - 35(m/s)\hat{\boldsymbol{\jmath}}\ln\frac{0.10kg}{1.00kg} \\
\vec{\boldsymbol{v}}_2 &= 65.890478(m/s)\hat{\boldsymbol{\jmath}} = 66(m/s)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

Next, we must find the change in position of the rocket over the burn time.

To do this, we will need a more specific expression of how M changes with time. Let $b = \frac{dM}{dt}$. Since the instantaneous burn rate b is constant, b equals to the average burn rate $\frac{\Delta M}{\Delta t} = \frac{-.90kg}{1.50s} = -.60kg/s$. Therefore, $M = M_1 + bt$.

If $\vec{\boldsymbol{r}}$ is the rocket's position vector, and $\vec{\boldsymbol{r}}_1 = \vec{\mathbf{0}}$ and $\vec{\boldsymbol{r}}_2$ are the positions at times t_1 and t_2, then we can find $\Delta\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1$ by integrating (2) twice:

$$\begin{aligned}
\int_{\vec{\boldsymbol{v}}_1}^{\vec{\boldsymbol{v}}} d\vec{\boldsymbol{v}} &= \vec{\boldsymbol{g}} \int_{t_1}^{t} dt + \vec{\boldsymbol{u}} \int_{M_1}^{M} \frac{dM}{M} \\
\vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_1 &= \vec{\boldsymbol{g}}(t - t_1) + \vec{\boldsymbol{u}} \ln \frac{M}{M_1} \\
\frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{v}} &= \vec{\boldsymbol{g}} t + \vec{\boldsymbol{u}} \ln \frac{M_1 + bt}{M_1} \\
d\vec{\boldsymbol{r}} &= \vec{\boldsymbol{g}} t \, dt + \vec{\boldsymbol{u}} \ln \frac{M_1 + bt}{M_1} \, dt \\
\int_{\vec{\boldsymbol{r}}_1}^{\vec{\boldsymbol{r}}_2} d\vec{\boldsymbol{r}} &= \vec{\boldsymbol{g}} \int_{t_1}^{t_2} t \, dt + \vec{\boldsymbol{u}} \int_{t_1}^{t_2} \ln\left(1 + \frac{bt}{M_1}\right) dt
\end{aligned}$$

For the last integral, make the substitution $w = 1 + \frac{bt}{M_1}$ (so $dw = \frac{b}{M_1} dt$):

$$\begin{aligned}
\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1 &= \frac{1}{2}\vec{\boldsymbol{g}}(t_2^2 - t_1^2) + \vec{\boldsymbol{u}} \int_{w_1}^{w_2} (\ln w) \left(\frac{M_1}{b} \, dw\right) \\
\vec{\boldsymbol{r}}_2 - \vec{\mathbf{0}} &= \frac{1}{2}\vec{\boldsymbol{g}}(t_2^2 - 0^2) + \frac{M_1 \vec{\boldsymbol{u}}}{b} \int_{w_1}^{w_2} \ln w \, dw
\end{aligned}$$

where $w_1 = 1 + \frac{bt_1}{M_1} = 1$ and $w_2 = 1 + \frac{bt_2}{M_1} = 0.10$. Using integration by parts, one can prove that $\int \ln w \, dw = w(\ln w - 1) + c$. Use $f(w) = \ln w$ and $g'(w) = 1$:

$$\begin{aligned}
\int \ln w \, dw = \int f g' \, dw &= fg - \int g f' \, dw \\
&= (\ln w) w - \int w \left(\frac{1}{w}\right) dw \\
&= w \ln w - \int dw \\
&= w \ln w - w + c \\
&= w(\ln w - 1) + c
\end{aligned}$$

Therefore:

$$\begin{aligned}
\vec{\boldsymbol{r}}_2 - \vec{\mathbf{0}} &= \frac{1}{2}\vec{\boldsymbol{g}}(t_2^2 - 0^2) + \frac{M_1 \vec{\boldsymbol{u}}}{b} (w(\ln w - 1))\Big|_{w_1}^{w_2} \\
\vec{\boldsymbol{r}}_2 &= \frac{1}{2}\vec{\boldsymbol{g}} t_2^2 + \frac{M_1 \vec{\boldsymbol{u}}}{b} (w_2(\ln w_2 - 1) - w_1(\ln w_1 - 1)) \\
\vec{\boldsymbol{r}}_2 &= \frac{1}{2}(-9.80 m/s^2)\hat{\boldsymbol{\jmath}}(1.50 s)^2 + \frac{1.00 kg(-35 m/s)\hat{\boldsymbol{\jmath}}}{-0.60 kg/s}(0.1(\ln 0.1 - 1) - 1(\ln 1 - 1)) \\
\vec{\boldsymbol{r}}_2 &= -(4.90 m)\hat{\boldsymbol{\jmath}}(2.25) + \frac{35 m}{0.60}(0.1(-2.3025851 - 1) - 1(0 - 1))\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}}_2 &= (28.043254 m)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

So, the rocket has height 28m at the end of the burn.

Immediately after the burn, the rocket has upward velocity, and so it will continue to rise until some later time t_3 when the velocity $\vec{\boldsymbol{v}}_3$ is $\vec{\mathbf{0}}$. Between times t_2 and t_3, the change in kinetic energy is

$$\Delta K = \frac{1}{2}M_2(v_3^2 - v_2^2) = -\frac{1}{2}M_2 v_2^2.$$

Ignoring air resistance, total energy $E = K+U$ is conserved, so $\Delta U = -\Delta K$ where $\Delta U = M_2 g \Delta r_y$ is the change in potential energy. Hence,

$$\begin{aligned}
\frac{1}{2}M_2 v_2^2 &= -\Delta K = \Delta U = M_2 g \Delta r_y \\
\frac{v_2^2}{2g} &= \Delta r_y \\
221.50791m &= \Delta r_y \\
249.55117m &= r_{2y} + \Delta r_y = r_{3y}
\end{aligned}$$

So, the maximum height of the rocket is 250m.

Space launches. For a rocket going into space, it is no longer a good approximation to assume that gravitational acceleration is constant. One should replace the gravitational force $M\vec{\boldsymbol{g}}$ with $-GMm_E\hat{\boldsymbol{r}}/r^2$ where G is the universal gravitational constant, m_E is Earth's mass, $\hat{\boldsymbol{r}}$ is the unit vector pointing from the center of the earth to the rocket, and r is the distance from the center of the earth to the rocket. This change to equation (1) makes it much more difficult to solve.

Besides non-constant gravitational acceleration, one must also take into account the spinning of Earth, as well as air resistance, making for a truly three-dimensional vector problem. (The reason NASA launches from Texas and Florida is that Earth spins faster there than in more northerly states.)