Forbidden local bases

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Basic terminology

- **Convention:** All spaces are Hausdorff.
- $\mathcal{C}(X, \mathbb{R})$ is the set of continuous functions from X to $\mathbb{R}$.
- $\mathrm{Aut}(X)$ is the group of homemomorphisms from X to X (the autohomeomorphism group).
- A space X is **homogeneous** if for all points p, q there exists $h \in \mathrm{Aut}(X)$ sending p to q.
- For compact X, the autohomeomorphisms of X are exactly the continuous permutations of X.

The big open problem

1. **Is every compact space a continuous image of a compact homogeneous space (CHS)?**
2. Every compact space is a continuous image of a 0-dimensional compact space.
3. (Milovich, 2007) Every CHS is a continuous image of an open subset of a path-connected CHS.
4. Is every compact space a continuous image of a 0-dimensional CHS?
5. Equivalent to (4), does every boolean algebra A extend to an ideally homogeneous boolean algebra B?
6. By "**ideally homogeneous**," I mean that for every two maximal ideals I, J of B, there is an automorphism h of B such that $h(I) = J$.

Some obstructions

Why doesn't it work to extend a boolean algebra to its completion? Why isn't a sufficiently large copower of a boolean algebra ideally homogeneous?

- A very weak form of completeness is the **countable separation property (CSP)**, which says that orthogonal countably generated ideals extend to orthogonal principal ideals.
- (Kunen, 1990) No infinite boolean algebra with the CSP is ideally homogeneous, nor are coproducts of such algebras.
- Actually, Kunen proved a more general topological statement: no infinite compact F-space is homogeneous, nor are products of such spaces.
- There are several other obstructions in the literature, stated in terms of topological cardinal functions, that prevent powers of various compact spaces from being homogeneous.

Some special cases of the big open problem

- Trivially, every **dyadic** compact X is by definition a continuous image of some 2^κ, which is a CHS.
- Less trivially, but well-known and not hard, every metrizable compactum is a continuous image of 2^ω.
- Is $\beta\mathbb{N}$ a continuous image of a CHS?
- Is $\beta\mathbb{N} \setminus \mathbb{N}$ a continuous image of a CHS?
- ((Clopen version of) Van Douwen's Problem, c. 1970) If D is discrete and $|D| > \mathfrak{c}$, is the one-point compactification $D \cup \{\infty\}$ a continuous image of a CHS? Equivalently, is there a CHS with a pairwise disjoint clopen family of size greater than $\mathfrak{c}$?
- (Maurice, 1964) $2^{\omega\cdot\omega}_{\mathrm{lex}}$ is a CHS with a pairwise disoint clopen family of size $\mathfrak{c}$, so if D is discrete and $|D| \leq \mathfrak{c}$, then $D \cup \{\infty\}$ a continuous image of a CHS.

A newly solved special case of the big open problem

1. A compact space X is **openly generated** iff there is a closed unbounded set $\mathcal{D}$ of countable subsets of $\mathcal{C}(X, \mathbb{R})$ such that for all $A \in \mathcal{D}$, the natural quotient map from X to X/A induced by A is open.
2. (Milovich, 2012) Every openly generated 0-dimensional compact space is a continuous image of an openly generated 0-dimensional CHS.
3. Equivalently, every boolean algebra with the Freese-Nation property extends to a boolean algebra an ideally homogeneous boolean algebra with the Freese-Nation property.
4. (Shapiro, 1976) For all $\kappa \geq \omega_2$, the Vietoris hyperspace $\mathrm{Exp}(2^\kappa)$ is openly generated but not dyadic.
5. (Shchepin, 1980) Every openly generated compactum is ccc.
6. Optimistic conjecture: I'm close to a modification of the proof of (2) that answers "yes" to the big open problem.

Cofinal types: another obstruction

- Quasi-orders P and Q have the same **cofinal type** iff they are isomorphic to cofinal suborders P', Q' of a common quasi-order R.
- For directed quasi-orders, P and Q have the same cofinal type iff there exist Tukey maps $f\colon P \to Q$ and $g\colon Q \to P$.
- $f\colon P \to Q$ is **Tukey** iff it maps all unbounded sets to unbounded sets (where "unbounded" means lacking an upper bound).

Cofinal types naturally emerge as an obstruction that we must climb over in proving that every openly generated compact 0-dimensional space is a continuous image of an openly generated 0-dimensional CHS.

The game

- Given compact X_0, Player I tries to build an inverse limit $X_\lambda = \varprojlim_{i<\lambda} X_i$ (with surjective continuous bonding maps) such that X_λ is a CHS.
- At each stage $i+1$, he builds X_{i+1}, $\pi_i \colon X_{i+1} \to X_i$, and $h_{i+1} \in \mathrm{Aut}(X_{i+1})$.
- Player I tries to ensure that for all $p, q \in X_\lambda$ there exists $A \subset \lambda$ unbounded such that $h_A = \varprojlim_{i \in A} h_{i+1} \in \mathrm{Aut}(X_\lambda)$ and $h_A(p) = q$.

Player II is a "diagonalizer" who trys to build $a_\lambda = \varprojlim_{i<\lambda} a_i$ and $b_\lambda = \varprojlim_{i<\lambda} b_i$ such that for all $i < \lambda$, $h_{i+1}(a_{i+1}) \neq b_{i+1}$.

First steps towards defeating the diagonalizer

- Choose $\lambda > w(X_0)$ such that $\diamondsuit(\lambda)$ holds and let Ξ be a $\diamondsuit$-sequence for all potential members of $\bigcup_{i<\lambda} X_i$.
- (With a little more work, $\diamondsuit(\lambda)$ can be weakened to $\lambda = 2^{<\lambda}$.)
- Whenever $\Xi(i) = (p_i, q_i) \in X_i^2$, build X_{i+1}, π_i, h_{i+1} such that
  - $w(X_{i+1}) = w(X_i)$;
  - $\pi_i^{-1}\{p_i\} = \{p_i\}$ and $\pi_i^{-1}\{q_i\} = \{q_i\}$;
  - $h_{i+1}(p_i) = q_i$.
- The above cannot be done if the neighborhood filters $\mathrm{Nbhd}(p_i, X_i)$ and $\mathrm{Nbhd}(q_i, X_i)$ have the different cofinal types (when ordered by $\supset$).
- The above can be done if X is homemorphic to $X \times 2^{\chi(X)}$. In this case, every point's neighborhood filter has the same cofinal type as $([\chi(X)]^{<\omega}, \subset)$.

The flatness conjecture

- We say a point in a space is **flat** it its neighborhood filter, ordered by $\supset$, has the same cofinal type as $([\kappa]^{<\omega}, \subset)$ for some κ.
- A point p is flat iff it has a local base $\mathcal{A}$ such that for every infinite $\mathcal{E} \subset \mathcal{A}$, p is not in the interior of $\bigcap \mathcal{E}$.
- **Conjecture.** In every CHS, all (equivalently, some) points are flat.
- (Milovich, 2008) Every known CHS has only flat points.
- (Milovich, 2008) All continuous images of openly generated compacta (including all dyadic compacta) have only flat points.

Special cases of the flatness conjecture

- **Conjecture.** In a CHS, no point's neighborhood filter has the cofinal type of $\omega \times \omega_1$.
- (Milovich, 2011) In every compact space, some point's neighborhood filters has cofinal type different from that of $\omega \times \omega_2$.
- The above result holds if we replace $\omega \times \omega_2$ with any product of a finite non-convex subset of the class of regular ordinals, like $\omega \times \omega_1 \times \omega_3$.
- In $X = \prod_{0 \leq i \leq n} 2^{\omega_i}_{\mathrm{lex}}$, all points' neighbhorhood filter have the cofinal type of $\prod_{0 \leq i \leq n} \omega_i$. But X as above is not homogeneous for $n \geq 1$ because for all $i \in [0, n]$, some point has π-character $\aleph_i$.

PCF theory applied to parametrized flatness

- Let $X = 2^{\aleph_\omega}$ and let X_δ be its G_δ modification.
- It is easy to show that the natural base $\mathcal{B}$ of X_δ is such that if $\mathcal{E} \in [\mathcal{B}]^{\mathfrak{c}^+}$, then $\bigcap \mathcal{E}$ has no subset in $\mathcal{B}$. Call this property $\mathfrak{c}^+$-**flatness**.
- (Kojman-Milovich-Spadaro, 2010) X_δ has an $\aleph_4$-flat base.
- (Soukup, 2010) If $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, then X_δ does not have an $\aleph_1$-flat base.
- (Kojman-Milovich-Spadaro, 2010) $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is consistent with $2^\omega = \aleph_\omega^\omega$, and the latter implies that X_δ has an $\aleph_1$-flat π-base.
- What happens in models of both GCH and $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$?