BRANCH PRODUCT RELATIONS

DAVID MILOVICH

Abstract. We define branch products relations, a new product construction. Branch product relations generalize direct products, lexicographic products, and ordered sums. We investigate criteria for various properties of relations to be preserved by branch products, focusing on branch products over trees. Among our nicer results are the preservation of poset dimension and various order-completeness properties by branch products over trees.

Introduction

Given a tree, what is a reasonable way to order its branches? Unless no ordering is desired, the nearly unanimous answer is "By first differences." More explicitly, this answer means two things should be done. First, certain subsets of the tree should be given linear orderings such that, given two distinct branches A and B, exactly one of these linear orderings will order $\min A \setminus B$ and $\min B \setminus A$. Second, A and B should be ordered as $\min A \setminus B$ and $\min B \setminus A$ are ordered. For example, a Souslin line is constructed from a Souslin tree by judiciously adding linear orders and then ordering the branches by first differences. For more about the relationship between linear orders and trees, see Todorčevic[2].

The goal of this paper is to generalize this way of ordering branches and to derive correspondingly general results. Instead of just trees with linear orders, we start with well-founded posets with relations. In this general setting, direct products, lexicographic products, and ordered sums all become special cases. We then specialize to trees with partial orders to obtain most of our results.

1. Preliminaries

Definition 1.1. Let $\langle X, \sqsubseteq\rangle$ be a nonempty poset. A *branch* of X is a maximal chain of X. Let $\mathscr{B}(X)$ denote the set of all branches of X. A *semibranch* is an initial segment of a branch. A semibranch is *proper* if it is not a branch. Let $\mathscr{S}(X)$ denote the set of all proper semibranches of X. For each $S \in \mathscr{S}(X)$, define the *fork* of S to be the set of minimal strict upper bounds of S, and denote the fork of S by $\mathscr{F}_X(S)$. For each $E \subseteq X$, let $\mathscr{B}_X(E)$ denote the set of branches of X that intersect E.

Remark 1.2. From the above definitions it is immediate that the every fork is an antichain; hence, the intersection of a fork and a chain contains at most one element.

Definition 1.3. By the previous remark, if C is a chain in X and $S \in \mathscr{S}(X)$ and $C \cap \mathscr{F}_X(S) \neq \emptyset$, then $C \cap \mathscr{F}_X(S)$ is a singleton, and we denote its element by $C@S$. For each $S \in \mathscr{S}(X)$, let π_S denote the map from $\mathscr{B}_X(\mathscr{F}_X(S))$ to $\mathscr{F}_X(S)$ given by $\pi_S(A) = A@S$ for all $A \in \mathscr{B}_X(\mathscr{F}_X(S))$.

Date: August 7, 2004.

We henceforth assume X is well-founded. Thus, $\mathscr{F}_X(S) \neq \emptyset$ for all $S \in \mathscr{S}(X)$. Moreover, all semibranches are well-ordered, allowing us to make the following definition.

Definition 1.4. For every $\alpha \in \mathrm{O}n$, we define S_α to be the unique semibranch R such that $R \subseteq S$ and the order type of R is the minimum of the order type of S and the order type of α. Define $h(S)$ to be the minimum ordinal α for which $S_\alpha = S$.

From this definition, we can immediately conclude the following proposition. The proof is a simple application of transfinite induction.

Proposition 1.5. *Let S be a semibranch and let $\alpha \in \mathrm{O}n$.*

1. *If $\alpha < h(S)$, then $S_{\alpha+1} = S \cup \{S@S_\alpha\}$.*
2. *If α is a limit ordinal, then $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$.*
3. *A branch B can be defined recursively by defining $B@B_\alpha$ in terms of B_α for each $\alpha < h(B)$. More precisely, given a function $f\colon \mathscr{S}(X) \to X$ such that $f(S) \in \mathscr{F}_X(S)$ for all $S \in \mathscr{S}(X)$, there is a unique branch B such that $B@B_\alpha = f(B_\alpha)$ for all $\alpha < h(B)$.*

Branch product relations over X are constructed by giving each fork of X a binary relation and then using these relations to induce a binary relation on $\mathscr{B}(X)$. Note that all relations are henceforth assumed to be binary. As a subset of X, a fork is simply referred to as a fork, but in the context of its relation, it is referred to as a *fork space*. Here one must be careful with terminology. For example, while a fork is always an antichain, a fork could be endowed with a linear order, so that the corresponding fork space is a chain. The formal definition of branch product relations is given below.

Definition 1.6. For each $S \in \mathscr{S}(X)$, let $\leq_S$ be a relation on $\mathscr{F}_X(S)$, which we call the *fork relation* of S. Thus, $\langle \mathscr{F}_X(S), \leq_S \rangle$ is a fork space. We denote the *branch product relation* on $\mathscr{B}(X)$ by $\leq$ and define it as follows. Let $A, B \in \mathscr{B}(X)$. Then $A \leq B$ if $A@S \leq_S B@S$ for all $S \in \mathscr{S}(X)$ for which both $A@S$ and $B@S$ exist. When the fork relations are not clear from the context, we use $\mathscr{B}\big(X, \langle \leq_S \rangle_{S \in \mathscr{S}(X)}\big)$ to denote the set $\mathscr{B}(X)$ together with the branch product relation induced by the fork relations $\langle \leq_S \rangle_{S \in \mathscr{S}(X)}$.

The motivation for the above definition is to generalize direct products, lexicographic products, and ordered sums. As demonstrated by the following examples, branch product relations achieve this generalization.

Example 1.7. Let $\langle X, \sqsubseteq \rangle = \big\langle \bigcup_{n<\omega} \{0,1\}^{\{0,\ldots,n\}}, \subseteq \big\rangle$. Then each fork has cardinality 2. We make each such fork into a fork space with order type 2. Then the branch product relation has the order type of the Cantor set.

Example 1.8. Generalizing the last example, let $\alpha \in \mathrm{O}n$ and let L_β be a nonempty chain for each $\beta < \alpha$. Let $\langle X, \sqsubseteq \rangle = \big\langle \bigcup_{\beta<\alpha} \prod_{\gamma\leq\beta} L_\gamma, \subseteq \big\rangle$. Then each fork is a copy of L_β for some $\beta < \alpha$, and we give it the corresponding fork relation. The branch product relation makes $\mathscr{B}(X)$ isomorphic to $\prod_{\beta<\alpha} L_\beta$ ordered by first differences; hence, the branch product relation generalizes the lexicographic product.

Example 1.9. Let $\alpha \in \mathrm{O}n$ and let X_β be a nonempty set with a relation R_β for each $\beta < \alpha$. Assume that these spaces are pairwise disjoint. Let $X = \bigcup_{\beta<\alpha} X_\beta$. For all $p, q \in X$, we declare $p \sqsubseteq q$ if $p = q$ or there exist $\beta, \gamma \in \mathrm{O}n$ such that $\gamma < \beta < \alpha$

and $p \in X_\gamma$ and $q \in X_\beta$. Then every fork is equal to X_β for some $\beta < \alpha$, and we give this fork the relation R_β. Then $\mathscr{B}(X)$ is isomorphic to $\prod_{\beta<\alpha} X_\beta$ with the direct product relation; hence, branch product relations generalize direct products.

Example 1.10. Let A be a nonempty set with a relation R. For each $a \in A$, let B_a be a nonempty set with a relation R_a. Assume that these sets are pairwise disjoint. Let $X = A \cup \bigcup_{a\in A} B_a$. For all $p, q \in X$, we declare $p \sqsubseteq q$ if $p = q$ or if $p \in A$ and $q \in B_p$. Then $\mathscr{S}(X) = \{\emptyset\} \cup \{\{a\} : a \in A\}$. Let $\leq_\emptyset = R$ and let $\leq_{\{a\}} = R_a$ for each $a \in A$. Then $\mathscr{B}(X) = \{\{a, b\} : a \in A \text{ and } b \in B_a\}$ and with the branch product relation $\mathscr{B}(X)$ is isomorphic to the ordered sum $\sum_{a\in A} B_a$; hence, branch product relations generalize the ordered sum.

2. Basic Properties

In this section we develop the more basic results about branch product relations. First, we characterize branch product relations in terms of relation-preserving maps.

Definition 2.1. Given sets A_1 and A_2 and relations R_1 and R_2 respectively on A_1 and A_2, we say that a map $f\colon A_1 \to A_2$ is *isotone* if xR_1y implies $f(x)R_2f(y)$ for all $x, y \in A_1$.

Theorem 2.2. *The branch product relation is the weakest relation on $\mathscr{B}(X)$ for which π_S is isotone for all $S \in \mathscr{S}(X)$.*

Proof. By definition of the branch product relation, π_S is clearly isotone for all $S \in \mathscr{S}(X)$. Suppose R is a relation on $\mathscr{B}(X)$ for which π_S is isotone for all $S \in \mathscr{S}(X)$. Further suppose $A, B \in \mathscr{B}(X)$ and ARB. Then, by isotonicity, $A@S \leq_S B@S$ for all S for which $A@S$ and $B@S$ exist; hence, $A \leq B$. Thus, R is stronger than $\leq$. □

We also note that branch products preserve very basic properties of relations.

Proposition 2.3. *The branch product relation on $\mathscr{B}(X)$ is reflexive if all fork relations of X are reflexive. Moreover, the branch product relation on $\mathscr{B}(X)$ is symmetric if all fork relations of X are symmetric, for the dual of $\mathscr{B}\big(X, \langle\leq_S\rangle_{S\in\mathscr{S}(X)}\big)$ is $\mathscr{B}\big(X, \langle\geq_S\rangle_{S\in\mathscr{S}(X)}\big)$.*

Proof. Let $A, B \in \mathscr{B}(X)$. Suppose all fork relations of X are reflexive. Then $A@S \leq_S A@S$ for all $S \in \mathscr{S}(X)$ for which $A@S$ exists; hence, $A \leq A$. The rest of the proposition is obvious. □

Definition 2.4. Given $n < \omega$, we say that a relation R is *n-acyclic* if, for all elements $a_0, \dots, a_{n-1}$ of the domain of R such that $a_0Ra_1R\cdots Ra_{n-1}Ra_0$, we have $a_0 = \cdots = a_{n-1}$.

Theorem 2.5. *Suppose $n < \omega$ and all fork relations of X are n-acyclic. Then the branch product relation on $\mathscr{B}(X)$ is n-acyclic.*

Proof. Suppose $A^{(0)}, \dots, A^{(n-1)} \in \mathscr{B}(X)$ and $A^{(0)} \leq A^{(1)} \leq \cdots \leq A^{(n-1)} \leq A^{(0)}$. Since $\bigcap_{m<n} A^{(m)}$ is chain, the union of all semibranches contained in this chain is a semibranch. Let S be this union. Further suppose there exist $i, j < n$ such that $A^{(i)} \neq A^{(j)}$. Then $S \subsetneq A^{(i)}$; hence, $S \in \mathscr{S}(X)$. Therefore, $A^{(m)}@S$ exists for all $m < n$. Hence, $A^{(0)}@S \leq_S A^{(1)}@S \leq_S \cdots \leq_S A^{(n-1)}@S \leq_S A^{(0)}@S$. By the n-acyclicity of $\leq_S$, we have $A^{(0)}@S = \cdots = A^{(n-1)}@S$. Hence, $S \cup \{A^{(0)}@S\}$

is a semibranch contained in $\bigcap_{m<n} A^{(m)}$, in contradiction with the definition of S. Therefore, $A^{(0)} = \cdots = A^{(n-1)}$. $\square$

Corollary 2.6. *Suppose all fork relations of X are antisymmetric. Then the branch product relation on $\mathscr{B}(X)$ is antisymmetric.*

Proof. Antisymmetry is equivalent to 2-acyclicity. $\square$

Corollary 2.7. *Suppose all fork relations of X are partial orders. Then the transitive closure of the branch product relation on $\mathscr{B}(X)$ is a partial order.*

Proof. Let $\leq^*$ denote the transitive closure of the branch product relation $\leq$. By Proposition 2.3, $\leq$ is reflexive; hence, $\leq^*$ is reflexive. Therefore, it suffices to prove that $\leq^*$ is antisymmetric. Suppose $A, B \in \mathscr{B}(X)$ and $A \leq^* B \leq^* A$. Then there exist $m, n < \omega$ such that

$$A = C^{(0)} \leq C^{(1)} \leq \cdots \leq C^{(m-1)} = B = C^{(m-1)} \leq C^{(m)} \leq \cdots \leq C^{(m+n-1)} = A$$

for some branches $C^{(0)}, \ldots, C^{(m+n-1)}$. Since every fork relation is a partial order, it is $(m+n)$-acyclic. Hence, $\leq$ is $(m+n)$-acyclic. Hence, $C^{(0)} = \cdots = C^{(m+n-1)}$. Hence, $A = B$. $\square$

In general, branch product relations are not transitive, even if all fork spaces are chains. To avoid this problem, we specialize from well-founded posets to trees. Recall that a tree is a poset in which every element's set of lower bounds is well-ordered. Henceforth, let W denote a nonempty tree.

Proposition 2.8. *No two forks of W intersect.*

Proof. Let S_1 and S_2 be proper semibranches. Suppose $p \in \mathscr{F}_W(S_1) \cap \mathscr{F}_W(S_2)$. Let P be the set of predecessors of p. Then P contains $S_1 \cup S_2$. Moreover, P is well-ordered because W is a tree. Therefore, S_1 and S_2 are initial segments of P. But by definition of fork, neither S_1 or S_2 can be a proper initial segment of P; hence, $S_1 = P = S_2$. $\square$

Definition 2.9. For any semibranch S and $a, b \in \mathscr{F}_W(S)$, we denote $a \leq_S b$ by $a \leq b$. Since no two forks intersect, this causes no ambiguity, provided the context determines whether we are relating elements of W or branches of W.

Proposition 2.10. *Suppose every fork relation of W is reflexive, and A and B are distinct branches of W. Then $A \leq B$ if and only if $A@(A \cap B) \leq B@(A \cap B)$.*

Proof. If $A \leq B$, then clearly $A@(A \cap B) \leq B@(A \cap B)$, as both $A@(A \cap B)$ and $B@(A \cap B)$ exist. Conversely, suppose $A@(A \cap B) \leq B@(A \cap B)$. Suppose $S \in \mathscr{S}(W)$ and $A@S$ and $B@S$ exist. Then $S \subseteq A \cap B$, for S is the set of predecessors of both $A@S$ and $B@S$. If $S \subsetneq A \cap B$, then $A@S, B@S \in A \cap B$; hence, $A@S = B@S$; hence, $A@S \leq B@S$. If $S = A \cap B$, then $A@S \leq B@S$ by assumption. Therefore, $A \leq B$. $\square$

Paraphrasing Proposition 2.10, when all fork relations are reflexive, $\mathscr{B}(W)$ is ordered by "first differences."

Definition 2.11. Let $\mathcal{M}$ be a class of objects of the form $\langle A, R \rangle$ where R is a relation on A. We say that $\mathcal{M}$ is *closed under tree branch products* if, given any nonempty tree W with all its fork spaces in $\mathcal{M}$, we have $\langle \mathscr{B}(W), \leq \rangle \in \mathcal{M}$.

Given this definition, we investigate which classes are closed under tree branch products. For basic terminology and notation of order theory and lattice theory, see Davey and Priestley[1].

Remark 2.12. Although we focus on order-theoretic applications of branch products, there certainly are other interesting aspects of branch products. For example, one might investigate which finite graphs are irreducible with respect to branch products or which are irreducible with respect to tree branch products.

Theorem 2.13. *The class of posets is closed under tree branch products.*

Proof. Assume that all fork spaces of W are posets. By Proposition 2.3 and Corollary 2.6, it suffices to prove $\mathscr{B}(W)$ is transitive. Let $A, B, C \in \mathscr{B}(W)$ and let $A \leq B \leq C$. If $A = B$ or $B = C$, then $A \leq C$. Suppose $A < B$ and $B < C$. Let $S = A \cap B \cap C$. Then, since $A \neq B$, we have $S \in \mathscr{S}(W)$. Also, $A@S \leq B@S \leq C@S$. If $A@S = C@S$, then $A@S = B@S = C@S \in A \cap B \cap C = S$, which is absurd. Therefore, $A@S < C@S$. Since $A@S \neq C@S$, we have $S = A \cap C$; hence, $A@(A \cap C) < C@(A \cap C)$; hence, $A < C$. □

Theorem 2.14. *The classes of chains and dense chains are each closed under tree branch products.*

Proof. Assume all forks spaces of W are chains. By Theorem 2.13, $\mathscr{B}(W)$ is a poset. Let A and B be distinct elements of $\mathscr{B}(W)$. Then $A@(A \cap B) \neq B@(A \cap B)$; hence, $A@(A \cap B) < B@(A \cap B)$ or $A@(A \cap B) > B@(A \cap B)$; hence, $A < B$ or $A > B$. Suppose $A < B$ and the fork space $\mathscr{F}_W(A \cap B)$ is a dense chain. Then there exists $c \in \mathscr{F}_W(A \cap B)$ such that $A@(A \cap B) < c < B@(A \cap B)$; hence, $A < C < B$ for every branch C that contains $(A \cap B) \cup \{c\}$. □

Definition 2.15. Given a poset $\langle P, \leq \rangle$, we define its *dimension* to be the smallest cardinal κ for which there exist κ-many linear extensions $\{\langle P, \preceq_\alpha \rangle : \alpha < \kappa\}$ of $\langle P, \leq \rangle$ such that $\leq = \bigcap_{\alpha<\kappa} \preceq_\alpha$. The dimension of $\langle P, \leq \rangle$ is denoted by $\dim\langle P, \leq \rangle$, or $\dim P$ when there is no ambiguity.

Theorem 2.16. *Suppose every fork space of W is a poset. Then we have*

$$\dim \mathscr{B}(W) = \sup\{\dim\langle \mathscr{F}_W(S), \leq_S \rangle : S \in \mathscr{S}(W)\}.$$

Proof. Let $\kappa = \sup\{\dim\langle \mathscr{F}_W(S), \leq_S \rangle : S \in \mathscr{S}(W)\}$. Then, for each $S \in \mathscr{S}(W)$, there exists $\langle \preceq_{S,\alpha} \rangle_{\alpha<\kappa}$ such that $\leq_S = \bigcap_{\alpha<\kappa} \preceq_{S,\alpha}$ and $\langle \mathscr{F}_W(S), \preceq_{S,\alpha} \rangle$ is a linear extension of $\langle \mathscr{F}_W(S), \leq_S \rangle$ for all $\alpha < \kappa$. Let A, B be distinct branches of W, and let $R = A \cap B$. By Proposition 2.10, we have $A \leq B$ in $\mathscr{B}\big(W, \langle \leq_S \rangle_{S\in\mathscr{S}(W)}\big)$ if and only if $A@R \leq_R B@R$, which is true if and only if $A@R \preceq_{R,\alpha} B@R$ for all $\alpha < \kappa$. Again by Proposition 2.10, if $\alpha < \kappa$, then $A@R \preceq_{R,\alpha} B@R$ if and only if $A \leq B$ in $\mathscr{B}\big(W, \langle \preceq_{S,\alpha} \rangle_{S\in\mathscr{S}(W)}\big)$. Thus, $A \leq B$ in $\mathscr{B}\big(W, \langle \leq_S \rangle_{S\in\mathscr{S}(W)}\big)$ if and only if $A \leq B$ in $\mathscr{B}\big(W, \langle \preceq_{S,\alpha} \rangle_{S\in\mathscr{S}(W)}\big)$ for all $\alpha < \kappa$. Clearly, $\mathscr{B}\big(W, \langle \preceq_{S,\alpha} \rangle_{S\in\mathscr{S}(W)}\big)$ is an extension of $\mathscr{B}\big(W, \langle \leq_S \rangle_{S\in\mathscr{S}(W)}\big)$. Moreover, by Theorem 2.14, $\mathscr{B}\big(W, \langle \preceq_{S,\alpha} \rangle_{S\in\mathscr{S}(W)}\big)$ is a linear extension. Therefore, $\dim \mathscr{B}\big(W, \langle \leq_S \rangle_{S\in\mathscr{S}(W)}\big) \leq \kappa$.

Let $T \in \mathscr{S}(W)$. Then we can order-embed $\langle \mathscr{F}_W(T), \leq_T \rangle$ in $\mathscr{B}\big(W, \langle \leq_S \rangle_{S\in\mathscr{S}(W)}\big)$ by mapping each $p \in \mathscr{F}_W(T)$ to a branch containing $T \cup \{p\}$. Therefore, we have

$$\dim\langle \mathscr{F}_W(T), \leq_T \rangle \leq \dim \mathscr{B}\big(W, \langle \leq_S \rangle_{S\in\mathscr{S}(W)}\big).$$

Thus, $\kappa \leq \dim \mathscr{B}\big(W, \langle \leq_S \rangle_{S\in\mathscr{S}(W)}\big) \leq \kappa$. □

Corollary 2.17. *For every cardinal κ, the class of posets of dimension κ is closed under tree branch products.*

3. Boundedness and Completeness

Tree branch products behave very well with respect to boundedness and completeness properties of posets. We first prove this statement in very general terms and then derive more specific results as corollaries.

Definition 3.1. Given a class of posets $\mathcal{M}$, we say that a poset P is $\mathcal{M}$-*bounded* ($\mathcal{M}$-*complete*) if every set in $\mathcal{M} \cap \mathcal{P}(P)$ has an upper bound (least upper bound) in P. We say that a class of posets $\mathcal{M}$ is *full* if the following conditions hold:

(1) if $P \in \mathcal{M}$ and f is an isotone map from P to some other poset, then $f(P) \in \mathcal{M}$;
(2) if $P \in \mathcal{M}$ and $Q \subseteq P$ and $x < y$ for all $x \in P \setminus Q$ and $y \in Q$, then $Q \in \mathcal{M}$.

Theorem 3.2. *Let $\mathcal{M}$ be a full class of posets. Then the classes of $\mathcal{M}$-bounded posets and $\mathcal{M}$-complete posets are each closed under tree branch products. Furthermore, if all fork spaces of W are $\mathcal{M}$-complete posets and $\mathcal{A} \in \mathcal{P}(\mathscr{B}(W)) \cap \mathcal{M}$, then, for all $\alpha < h\left(\bigvee \mathcal{A}\right)$, we have*

$$\left(\bigvee \mathcal{A}\right) @ \left(\bigvee \mathcal{A}\right)_\alpha = \bigvee \left\{ A @ A_\alpha : A \in \mathcal{A} \text{ and } A_\alpha = \left(\bigvee \mathcal{A}\right)_\alpha \right\}. \tag{3.1}$$

Indeed, (3.1) holds for all $\mathcal{A} \subseteq \mathscr{B}(W)$ for which $\bigvee \mathcal{A}$ exists.

Proof. Let all fork spaces of W be $\mathcal{M}$-bounded posets and let $\mathcal{A} \in \mathcal{P}(\mathscr{B}(W)) \cap \mathcal{M}$. By Proposition 1.5, we can define a branch B in W by defining $B@B_\alpha$ in terms of B_α for every $\alpha < h(B)$. Suppose B_α has been defined, $B_\alpha \in \mathscr{S}(W)$, and, for all $\beta < \alpha$, we have $E_\beta \in \mathcal{M}$ and $B@B_\beta$ is an upper bound of E_β, where $E_\beta = \{A@A_\beta : A \in \mathcal{A} \text{ and } A_\beta = B_\beta\}$. Further suppose that $B@B_\beta = \bigvee E_\beta$ for all $\beta < \alpha$ such that $\mathscr{F}_W(B_\beta)$ is $\mathcal{M}$-complete. Provided $E_\alpha \in \mathcal{M}$, we may choose $B@B_\alpha$ such that it is an upper bound of E_α and is the least upper bound of E_α if $\mathscr{F}_W(B_\alpha)$ is $\mathcal{M}$-complete.

Let us show this provision is satisfied. Set

$$\mathcal{A}' = \mathscr{B}_W(\mathscr{F}_W(B_\alpha)) \cap \mathcal{A} = \{A \in \mathcal{A} : A_\alpha = B_\alpha\}.$$

Suppose $A \in \mathcal{A} \setminus \mathcal{A}'$ and $A' \in \mathcal{A}'$. Then there exists $\beta < \alpha$ such that $A_\beta = A'_\beta$ and $A@A_\beta \neq A'@A'_\beta$. Since $A' \in \mathcal{A}'$, we have $A_\beta = A'_\beta = B_\beta$ and $A'@A'_\beta = B@B_\beta$. Hence, $A@A_\beta \in E_\beta$; hence, $A@A_\beta \leq B@B_\beta = A'@A'_\beta$; hence, $A@A_\beta < A'@A'_\beta$; hence, $A < A'$. Thus, by fullness of $\mathcal{M}$, we have $\mathcal{A}' \in \mathcal{M}$. Moreover, since π_{B_α} is isotone, $E_\alpha = \pi_{B_\alpha}(\mathcal{A}') \in \mathcal{M}$.

By induction, for each $\alpha < h(B)$, we have $B@B_\alpha$ is an upper bound for E_α and is the least upper bound of E_α if $\mathscr{F}_W(B_\alpha)$ is $\mathcal{M}$-complete. Suppose there is an A in $\mathcal{A}$ such that $A \nleq B$. Then $A@(A \cap B) \nleq B@(A \cap B)$. Let $A_\alpha = B_\alpha = A \cap B$. Then $A@A_\alpha \in E_\alpha$; hence, $A@A_\alpha \leq B@B_\alpha$, which is absurd. Therefore, B is an upper bound of $\mathcal{A}$. Thus, $\mathscr{B}(W)$ is $\mathcal{M}$-bounded.

Suppose every fork space of W is $\mathcal{M}$-complete. Let us show $B = \bigvee \mathcal{A}$. Suppose not. Then $\mathcal{A}$ has an upper bound C satisfying $B \nleq C$; hence,

$$B@(B \cap C) \nleq C@(B \cap C).$$

Let $B_\beta = B \cap C$. If $E_\beta \neq \emptyset$, then, since C is an upper bound of $\mathcal{A}$, we have

$$C@(B \cap C) \geq \bigvee E_\beta = B@(B \cap C),$$

which is absurd. If $E_\beta = \emptyset$, then $B@(B \cap C) = \min \mathscr{F}_W(B_\beta) \leq C@(B \cap C)$, which is also absurd. Therefore, $B = \bigvee \mathcal{A}$. Thus, $\mathscr{B}(W)$ is $\mathcal{M}$-complete. Moreover, the definition of B clearly implies (3.1).

Now allow $\mathcal{A}$ to be an arbitrary subset of $\mathscr{B}(W)$. Suppose $\bigvee \mathcal{A}$ exists but (3.1) fails. Let α be the least ordinal such that (3.1) fails. Define E by

$$E = \left\{ A@A_\alpha : A \in \mathcal{A} \text{ and } A_\alpha = \left(\bigvee \mathcal{A}\right)_\alpha \right\}.$$

Then $(\bigvee \mathcal{A})@(\bigvee \mathcal{A})_\alpha$ is an upper bound of E. Since (3.1) fails, we may choose $p \in \mathscr{F}_W((\bigvee \mathcal{A})_\alpha)$ such that p is an upper bound of E but $(\bigvee \mathcal{A})@(\bigvee \mathcal{A})_\alpha \not\leq p$. Let C be an arbitrary branch containing p. Then $C \cap \bigvee \mathcal{A} = (\bigvee \mathcal{A})_\alpha$; hence, $C \not\leq \bigvee \mathcal{A}$; hence, C is not an upper bound of $\mathcal{A}$.

Choose $D \in \mathcal{A}$ such that $D \not\leq C$. Let $D_\beta = C_\beta = D \cap C$. Then $D@D_\beta \not\leq C@C_\beta$. If $\beta < \alpha$, then $D_\beta = (\bigvee \mathcal{A})_\beta$ and

$$C@C_\beta = \left(\bigvee \mathcal{A}\right) @ \left(\bigvee \mathcal{A}\right)_\beta = \bigvee \left\{ A@A_\beta : A \in \mathcal{A} \text{ and } A_\beta = \left(\bigvee \mathcal{A}\right)_\beta \right\} \geq D@D_\beta,$$

which is absurd. If $\beta = \alpha$, then $D@D_\beta \in E$ and $C@C_\beta$ is p, an upper bound of E, which is absurd. Therefore, $\beta > \alpha$. Hence, $D \cap (\bigvee \mathcal{A}) = C \cap (\bigvee \mathcal{A}) = (\bigvee \mathcal{A})_\alpha$; hence, $D_\alpha = (\bigvee \mathcal{A})_\alpha$; hence, $p = C@C_\alpha = D@D_\alpha \in E$. Thus, (3.1) holds for all $\alpha < h(\bigvee \mathcal{A})$. □

Definition 3.3. Let κ be a cardinal and let P be a poset. We say P is a *κ-complete lattice* if, for all $E \subseteq P$ satisfying $|E| < \kappa$, the set E has a supremum and an infimum in P. We say P is *κ-complete poset* if, for all empty or directed subsets E of P satisfying $|E| < \kappa$, the set E has a supremum in L. We say P is *κ-directed* if, for all $E \subseteq P$ satisfying $|E| < \kappa$, the set E has an upper bound in P.

Corollary 3.4. *Let κ be a cardinal. Then the classes of bounded lattices, complete lattices, κ-complete lattices, complete posets, κ-complete posets, directed posets, bounded posets, and κ-directed posets are each closed under tree branch products.*

Proof. First we reduce the number the cases we need to consider.

(1) The bounded lattices are exactly the $\aleph_0$-complete lattices.
(2) The directed posets are exactly the $\aleph_0$-directed posets.
(3) All complete lattices are $|\mathscr{B}(W)|^+$-complete lattices and $\mathscr{B}(W)$ is a complete lattice if it is a $|\mathscr{B}(W)|^+$-complete lattice.
(4) All complete posets are $|\mathscr{B}(W)|^+$-complete posets and $\mathscr{B}(W)$ is a complete poset if it is a $|\mathscr{B}(W)|^+$-complete poset.
(5) All bounded posets and their duals are $|\mathscr{B}(W)|^+$-directed, and $\mathscr{B}(W)$ is bounded if it and its dual are $|\mathscr{B}(W)|^+$-directed.

Thus, it suffices to show that the classes of κ-complete lattices, κ-complete posets, and κ-directed posets are each closed under tree branch products. Let $\mathcal{M}$ be the class of posets of size less than κ and let $\mathcal{N}$ be the class of empty or directed posets of size less than κ. Then $\mathcal{M}$ and $\mathcal{N}$ are full. Thus, if every fork space of W is a κ-complete lattice, then every fork space and its dual are $\mathcal{M}$-complete; whence, $\mathscr{B}(W)$ and its dual are $\mathcal{M}$-complete; whence, $\mathscr{B}(W)$ is a κ-complete lattice.

Similarly, if every fork space of W is a κ-complete poset, then every fork space is $\mathcal{N}$-complete; whence, $\mathscr{B}(W)$ is $\mathcal{N}$-complete; whence, $\mathscr{B}(W)$ is a κ-complete poset. Finally, suppose every fork space of W is a κ-directed poset. Then every fork space is $\mathcal{M}$-bounded; hence, $\mathscr{B}(W)$ is $\mathcal{M}$-bounded; hence, $\mathscr{B}(W)$ is a κ-directed poset. □

Corollary 3.5. *For each cardinal κ, the classes of bounded chains, complete chains, and κ-complete chains are each closed under tree branch products.*

The boundedness requirement for lattices in Corollary 3.4 cannot be eliminated, as shown below.

Example 3.6. *The class of lattices is not closed under tree branch products.* Let $\mathscr{F}_W(\emptyset) = \{a, b, c, d\}$ with $a < b < d > c > a$ and $b \not\leq c \not\leq b$. Let $\mathscr{F}_W(\{a\})$ be ω with its canonical ordering. Let W have no other forks. Then $\{b\}, \{c\} \in \mathscr{B}(W)$ but $\{\{b\}, \{c\}\}$ has no infimum in $\mathscr{B}(W)$, despite all forks in W being lattices.

4. Lattice Properties

The previous section shows that tree branch products behave very well with respect to order properties, and very well with respect to completeness. Unfortunately they do not behave nearly as well with respect to lattice properties of a more algebraic nature, as the next example shows.

Example 4.1. *The classes of boolean lattices, bounded distributive lattices, and bounded modular lattices are each not closed under tree branch products.* It suffices to exhibit a tree W such that all its fork spaces are boolean lattices but $\mathscr{B}(W)$ is not modular. Let $\mathscr{F}_W(\emptyset) = \{a, b, c, d\}$ with $a < b < d > c > a$ and $b \not\leq c \not\leq b$. Let $\mathscr{F}_W(\{b\}) = \{e, f\}$ with $e < f$. Let W have no other forks. Then we have

$$\mathscr{B}(W) = \{\{a\}, \{b, e\}, \{b, f\}, \{c\}, \{d\}\}.$$

Moreover, we have $\{b, e\} < \{b, f\}$ and

$$\{b, e\} \vee (\{c\} \wedge \{b, f\}) = \{b, e\} < \{b, f\} = (\{b, e\} \vee \{c\}) \wedge \{b, f\},$$

in violation of modularity.

The above example is a strong negative algebraic result, but we can show a small positive algebraic result.

Theorem 4.2. *The classes of bounded join-semidistributive lattices and bounded meet-semidistributive lattices are each closed under tree branch products.*

Proof. We prove the join-semidistributive case by contradiction. Duality handles the other case.

Suppose all the fork spaces of W are bounded join-semidistributive lattices. Then $\mathscr{B}(W)$ is a bounded lattice. Suppose $\mathscr{B}(W)$ is not join-semidistributive. Then there exist $A, B, C, D \in \mathscr{B}(W)$ such that $D = A \vee B = A \vee C$ but $D \neq A \vee (B \wedge C)$. Let $E = A \vee (B \wedge C)$. Then $D > E$. Let $D_\alpha = E_\alpha = D \cap E$. Then $D@D_\alpha > E@E_\alpha$.

Suppose $A_\alpha = B_\alpha = C_\alpha = D_\alpha$. Then, by (3.1) and its dual,

$$D@D_\alpha = A@A_\alpha \vee B@B_\alpha = A@A_\alpha \vee C@C_\alpha \text{ and}$$
$$E@E_\alpha = A@A_\alpha \vee (B@B_\alpha \wedge C@C_\alpha).$$

Since $\langle \mathscr{F}_W(E_\alpha), \leq \rangle$ is a join-semidistributive lattice, $D@D_\alpha = E@E_\alpha$, which is absurd. Therefore, $A_\alpha \neq D_\alpha$ or $B_\alpha \neq D_\alpha$ or $C_\alpha \neq D_\alpha$. Suppose $A_\alpha = D_\alpha \neq B_\alpha$.

Then, by (3.1), $A@A_\alpha = D@D_\alpha > E@E_\alpha$. Consequently, since $E \geq A$ and $E_\alpha = D_\alpha = A_\alpha$, we have $A@A_\alpha > E@E_\alpha \geq A@A_\alpha$, which is absurd. Likewise, $A_\alpha = D_\alpha \neq C_\alpha$ implies a contradiction. Therefore, $A_\alpha \neq D_\alpha$. Then (3.1) implies that $D@A_\beta > A@A_\beta$ for some $\beta < \alpha$.

Suppose $B_\alpha = D_\alpha$. Then $B@A_\beta = D@A_\beta > A@A_\beta$ for some $\beta < \alpha$; hence, $B > A$; hence, $D = A \vee B = B$. Since $B = D$ and $D = A \vee C \geq C$, we have $B \geq C$. Therefore,

$$D = A \vee C = A \vee (B \wedge C) = E,$$

which is absurd. Therefore, $A_\alpha \neq D_\alpha \neq B_\alpha$; hence, $D@D_\alpha = \min \mathscr{F}_W(D_\alpha)$ by (3.1). But then $D@D_\alpha \leq E@E_\alpha$, which is absurd. Therefore, $\mathscr{B}(W)$ is join-semidistributive. □

Tree branch products also almost preserve complementarity.

Proposition 4.3. *Suppose all the fork spaces of W are complemented (and hence bounded) lattices. Then $\mathscr{B}(W)$ is a bounded lattice and is complemented if the minimum and maximum of $\mathscr{F}_W(\emptyset)$ (with respect to the fork relation) are each contained in only one branch.*

Proof. Let $A \in \mathscr{B}(W)$. Let B be a branch containing the complement of $A@\emptyset$ in $\mathscr{F}_W(\emptyset)$. Then

$$(A \vee B)@\emptyset = \max \mathscr{F}_W(\emptyset) \text{ and } (A \wedge B)@\emptyset = \min \mathscr{F}_W(\emptyset)$$

by (3.1) and its dual. Therefore, $A \vee B = \max \mathscr{B}(W)$ and $A \wedge B = \min \mathscr{B}(W)$. □

Next we consider the classes of algebraic lattices and continuous lattices. Neither class is closed under tree branch products. Interestingly, the class of complete weakly atomic lattices, which contains the algebraic lattices, and the smaller class of strongly algebraic lattices (see Definition 4.7) are each closed under tree branch products. Moreover, when we restrict from lattices to chains, weak atomicity, algebraicity, and strong algebraicity coincide. There is also a subclass of the continuous lattices which is closed under tree branch products, and this subclass is not a proper subclass when restricted to chains. We call this subclass the class of everywhere weakly compact lattices (see Definition 4.7). Let us prove these assertions.

Theorem 4.4. *The class of bounded weakly atomic lattices is closed under tree branch products.*

Proof. Suppose all fork spaces of W are bounded weakly atomic lattices. Also suppose $A, B \in \mathscr{B}(W)$ and $A < B$. Then $A@(A \cap B) < B@(A \cap B)$. Since $\mathscr{F}_W(A \cap B)$ is weakly atomic, there exist $c, d \in \mathscr{F}_W(A \cap B)$ such that

$$A@(A \cap B) \leq c < d \leq B@(A \cap B)$$

and d is an immediate successor of c in $\mathscr{F}_W(A \cap B)$. Let $\alpha \in \mathrm{O}n$ satisfy $A_\alpha = A \cap B$. Then recursively define $C \in \mathscr{B}(W)$ by $C_{\alpha+1} = A_\alpha \cup \{c\}$ and $C@C_\beta = \max \mathscr{F}_W(C_\beta)$ for all $\beta \geq \alpha + 1$ such that $C_\beta \in \mathscr{S}(W)$. Define $D \in \mathscr{B}(W)$ by $D_{\alpha+1} = A_\alpha \cup \{d\}$ and $D@D_\beta = \min \mathscr{F}_W(D_\beta)$ for all $\beta \geq \alpha + 1$ such that $D_\beta \in \mathscr{S}(W)$. Then $A \leq C < D \leq B$ and D is an immediate successor of C. □

Corollary 4.5. *The class of complete weakly atomic lattices is closed under tree branch products.*

Example 4.6. *The classes of algebraic lattices and continuous lattices are each not closed under tree branch products.* Let $\mathscr{F}_W(\emptyset) = \bigcup_{n \leq \omega} \{a_n, b_n\}$. Let

$$a_{n+1} > a_n < b_n < b_{n+1} \not\leq a_\omega \not\leq b_n$$

for all $n < \omega$. Also, let $a_\omega = \bigvee \{a_n : n < \omega\}$ and $b_\omega = \bigvee \{b_n : n < \omega\}$ and $a_\omega < b_\omega$. Also, let $\mathscr{F}_W(\{a_\omega\}) = \{c, d\}$ and $c < d$. Finally, let W have no other forks. Then all fork spaces of W are algebraic, and hence continuous. But $\mathscr{B}(W)$ is not continuous, and hence not algebraic: $\{a_\omega, d\}$ is join-irreducible; hence, continuity of $\mathscr{B}(W)$ implies $\{a_\omega, d\}$ is compact. But $\{a_\omega, d\} < \bigvee \{\{b_n\} : n < \omega\}$ and $\{a_\omega, d\} \not\leq \{b_n\}$ for all $n < \omega$; hence, $\{a_\omega, d\}$ is not compact; hence, $\mathscr{B}(W)$ is not continuous.

Definition 4.7. Let L be a complete lattice, let p be an element of L, and let κ and λ be cardinals.

1. We say that p is *κ-compact* if, for every $A \subseteq L$ such that $p \leq \bigvee A$, there is a subset B of A such that $|B| < \kappa$ and $p \leq \bigvee B$. We abbreviate $\aleph_0$-compact by compact, in agreement with the usual definition of compact.
2. We say that p is *weakly κ-compact* if, for every $A \subseteq L$ such that $p < \bigvee A$, there is a subset B of A such that $|B| < \kappa$ and $p \leq \bigvee B$. We abbreviate weakly $\aleph_0$-compact by weakly compact.
3. We say that p is *strictly κ-compact* if, for every $A \subseteq L$ such that $p < \bigvee A$, there is a subset B of A such that $|B| < \kappa$ and $p < \bigvee B$. We abbreviate strictly $\aleph_0$-compact by strictly compact.
4. We say that L is *everywhere* κ-compact (*everywhere* weakly κ-compact, *everywhere* strictly κ-compact) if all its elements are κ-compact (weakly κ-compact, strictly κ-compact).
5. We say that L is *κ-algebraic* if all its elements are joins of κ-compact elements. We abbreviate $\aleph_0$-algebraic by algebraic, in agreement with the usual definition of algebraic.
6. We say that L is *κ-strongly* λ-algebraic if L is λ-algebraic and all its elements are strictly κ-compact. We abbreviate $\aleph_0$-strongly $\aleph_0$-algebraic by strongly algebraic.

Proposition 4.8. *Every complete chain is everywhere strictly* 2*-compact. Moreover, every weakly atomic complete chain is* 2*-strongly* 2*-algebraic, and hence algebraic.*

Proof. Let C be a complete chain. Let $a \in C$, let $B \subseteq C$, and let $a < \bigvee B$. Then there exists $b \in B$ such that $a < b$; hence, a is strictly 2-compact. Suppose C is also weakly atomic. Let D be the set of elements of C that have an immediate predecessor. Then every element of D is 2-compact. Let $E = \{d \in D : d \leq a\}$. Then $\bigvee E \leq a$. Suppose $\bigvee E < a$. Then, by weak atomicity of C, there exist $p, q \in C$ such that $\bigvee E \leq p < q \leq a$ and p is the immediate predecessor of q. Therefore, $\bigvee E < q \in E$, which is absurd. Therefore, $\bigvee E = a$; hence, a is a join of 2-compact elements. □

Proposition 4.9. *Let κ be a cardinal greater than* 1*. Then*

1. *strict κ-compactness implies weak κ-compactness,*
2. *weak κ-compactness implies strict $(\kappa + 1)$-compactness,*
3. *κ-compactness implies weak κ-compactness, and*
4. *strict κ-compactness does not imply κ-compactness.*

Proof. (1) and (3) are trivial. To prove (4), simply note that every non-compact element of a complete chain is strictly 2-compact by Proposition 4.8. To prove (2), let L be a complete lattice, let p be a weakly κ-compact element of L, and let $A \subseteq L$ satisfy $p < \bigvee A$. Then there exists $B \subseteq A$ such that $|B| < \kappa$ and $p \leq \bigvee B$. If $p < \bigvee B$, then we are done. If $p = \bigvee B$, then $\bigvee B < \bigvee A$; hence, there exists $a \in A$ such that $a \nleq \bigvee B$; hence, $p = \bigvee B < \bigvee (B \cup \{a\})$; hence, p is strictly $(\kappa+1)$-compact. □

Proposition 4.10. *Every complete chain, which is automatically a continuous lattice, is also everywhere weakly compact.*

Proof. By Proposition 4.8, every complete chain is everywhere 2-strictly compact. But 2-strict compactness implies weak 2-compactness, which implies weak compactness. □

Proposition 4.11. *Let L be a complete lattice and let $p \in L$. Then p is weakly compact if and only if $p \ll q$ for all $q > p$. Moreover, if L is everywhere weakly compact, then L is continuous.*

Proof. Let p be weakly compact and $q \in L$ and $A \subseteq L$ and $p < q \leq \bigvee A$. Then A has a finite subset B such that $p \leq \bigvee B$; hence, $p \ll q$ whenever $p < q$. Conversely, suppose $p \ll q$ for all $q > p$. Let $A \subseteq L$ satisfy $p < \bigvee A$. Then $p \ll \bigvee A$; hence, A has a finite subset B such that $p \leq \bigvee B$; hence, p is weakly compact. This proves the first part of the proposition.

Suppose L is everywhere weakly compact. Then

$$p \geq \bigvee \{q : q < p\} = \bigvee \{q : q \ll p\}.$$

We must show $p = \bigvee \{q : q \ll p\}$. Suppose $p > \bigvee \{q : q \ll p\} = \bigvee \{q : q < p\}$. Then let $A \subseteq L$ satisfy $p \leq \bigvee A$. Then $p < \bigvee A$ implies A has a finite subset B such that $p \leq \bigvee B$. On the other hand, $p = \bigvee A$ implies $p \in A$. Therefore, $p \ll p$. Hence, $p = \bigvee \{q : q \ll p\}$. Therefore, L is continuous. □

Theorem 4.12. *For each cardinal κ, the class of everywhere strictly κ-compact lattices is closed under tree branch products.*

Proof. Let all the fork spaces of W be everywhere strictly κ-compact lattices. Then $\mathscr{B}(W)$ is a complete lattice by Corollary 3.4. Let $A, B \in \mathscr{B}(W)$, let $\mathcal{C} \subseteq \mathscr{B}(W)$, and let $A < \bigvee \mathcal{C} = B$. Let $B_\alpha = A \cap B$. Then $A@A_\alpha < B@B_\alpha$; hence,

$$B@B_\alpha \neq \min \mathscr{F}_W(B_\alpha)\,.$$

Therefore, by (3.1), there exists $C \in \mathcal{C}$ such that $C_\alpha = B_\alpha$. Let

$$\mathcal{D} = \{C \in \mathcal{C} : C_\alpha = B_\alpha\}.$$

Then, by (3.1),

$$A@A_\alpha < B@B_\alpha = \bigvee \{D@D_\alpha : D \in \mathcal{D}\}.$$

Since $A@A_\alpha$ is strictly κ-compact in $\mathscr{F}_W(A_\alpha)$, there exist $\mathcal{E} \subseteq \mathcal{D}$ such that $|\mathcal{E}| < \kappa$ and

$$A@A_\alpha < \bigvee \{E@E_\alpha : E \in \mathcal{E}\}.$$

Since $E_\alpha = A_\alpha$ for all $E \in \mathcal{E}$, we have $A < \bigvee \mathcal{E}$ by (3.1). □

Corollary 4.13. *The class of everywhere weakly compact lattices is closed under tree branch products.*

Proof. Since $\aleph_0 + 1 = \aleph_0$, Proposition 4.9 implies that weak compactness is equivalent to strict compactness; apply Theorem 4.12. $\square$

Theorem 4.14. *Let κ and λ be cardinals satisfying $\kappa \leq \lambda$. Then the class of κ-strongly λ-algebraic lattices is closed under tree branch products.*

Proof. Let all the fork spaces in W be κ-strongly λ-algebraic lattices. Then, by Theorem 4.12, $\mathscr{B}(W)$ is an everywhere strictly κ-compact lattice. Let $B \in \mathscr{B}(W)$ and let $A = \bigvee \mathcal{A}$ where

$$\mathcal{A} = \{E \in \mathscr{B}(W) : E \leq B \text{ and } E \text{ is } \lambda\text{-compact}\}.$$

Then $A \leq B$. If $A = B$, then we're done. Suppose $A < B$. Let $A_\alpha = A \cap B$. Then $A@A_\alpha < B@B_\alpha$. Because $\mathscr{F}_W(A_\alpha)$ is λ-algebraic, there exists $c \in \mathscr{F}_W(A_\alpha)$ such that c is λ-compact, $c \nleq A@A_\alpha$, and $c \leq B@B_\alpha$. Recursively define $C \in \mathscr{B}(W)$ by $C_{\alpha+1} = B_\alpha \cup \{c\}$ and $C@C_\beta = \min \mathscr{F}_W(C_\beta)$ for all $\beta \geq \alpha+1$ such that $C_\beta \in \mathscr{S}(W)$. Then $C \leq B$. Also, $C \nleq A$; hence, $C \notin \mathcal{A}$; hence, C is not λ-compact.

To complete the proof, we will derive a contradiction by showing C is λ-compact. Let $\mathcal{D} \subseteq \mathscr{B}(W)$ satisfy $C \leq \bigvee \mathcal{D}$. Suppose $C < \bigvee \mathcal{D}$. Then, because C is strictly κ-compact, there exists $\mathcal{E} \subseteq \mathcal{D}$ such that $|\mathcal{E}| < \kappa \leq \lambda$ and $C < \bigvee \mathcal{E}$, as desired. Suppose $C = \bigvee \mathcal{D}$. Since $c \nleq A@A_\alpha$, we have $c \neq \min \mathscr{F}_W(C_\alpha)$; hence, by (3.1), there exists $D \in \mathcal{D}$ such that $D_\alpha = C_\alpha$. Let $\mathcal{G} = \{D \in \mathcal{D} : D_\alpha = C_\alpha\}$. Then (3.1) implies $c = \bigvee\{G@G_\alpha : G \in \mathcal{G}\}$. Since c is λ-compact, there exists $\mathcal{H} \subseteq \mathcal{G}$ such that $|\mathcal{H}| < \lambda$ and $c \leq \bigvee\{H@H_\alpha : H \in \mathcal{H}\}$. Therefore, $C \leq \bigvee \mathcal{H}$ as desired because $C@C_\beta = \min \mathscr{F}_W(C_\beta)$ for all $\beta \geq \alpha + 1$ for which $C_\beta \in \mathscr{S}(W)$. $\square$

Corollary 4.15. *The class of strongly algebraic lattices is closed under tree branch products.*

Acknowledgements

This research has been partially supported by the MIT Undergraduate Research Opportunities Program (UROP), in particular, the Meryl and Stewart Robertson UROP Fund. The author especially wishes to thank his UROP supervisor Prof. Jonathan Farley.

References

[1] Davey, B. A. and Priestley, H. A., *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, UK, 1990.

[2] Todorčevic, S., *Trees and linearly ordered sets*, in "Handbook of Set Theoretic Topology," K. Kunen and J. Vaughan (eds), North-Holland, Amsterdam, 1984.