A COMPACTNESS THEOREM FOR POSETS

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Theorem 1. *Let κ be an infinite cardinal and let $\mathcal{L}$ be a nonempty set of size less than κ such that $\langle \mathcal{L}, \supseteq \rangle$ is directed. Suppose that $\leq$ is a relation on $\bigcup \mathcal{L}$ that partially orders each $L \in \mathcal{L}$. Also suppose that, for each $L \in \mathcal{L}$, every empty or $\leq$-directed subset of L of size less than κ has a least upper bound with respect to $\langle L, \leq \rangle$. Finally, suppose that there exists an $L_0 \in \mathcal{L}$ such that κ cannot be order-embedded in $\langle L_0, \leq \rangle$. Then every $\leq$-directed subset of $\bigcap \mathcal{L}$ of size less than κ has a least upper bound with respect to $\langle \bigcap \mathcal{L}, \leq \rangle$. In particular, $\bigcap \mathcal{L} \neq \emptyset$.*

Proof. Let $E \subseteq \bigcap \mathcal{L}$ be such that $|E| < \kappa$ and E is directed. To construct the supremum of E in $\bigcap \mathcal{L}$, let us first recursively define a particular indexed set of points $\langle x_L^\alpha : \alpha < \kappa,\ L \in \mathcal{L} \rangle$ and simultaneously prove that, for all $\eta < \kappa$ and $\beta \leq \eta$ and $L \in \mathcal{L}$, we have

(1) $x_L^\beta \in L$,
(2) $x_L^\gamma \leq x_L^\beta$ if $\gamma \leq \beta$, and
(3) $x_L^\beta \leq x_M^\beta$ if $L \supseteq M \in \mathcal{L}$.

First, for each $L \in \mathcal{L}$, let x_L^0 be the least upper bound of E in L. Suppose $L, M \in \mathcal{L}$ and $L \supseteq M$. Then the least upper bound of E in M is an upper bound of E in L; hence, $x_L^0 \leq x_M^0$. Also, $x_L^0 \in L$ is clear. Thus, (1)-(3) hold when $\eta = 0$.

Now suppose $0 < \alpha < \kappa$ and (1)-(3) hold for all $\eta < \alpha$. Let us define $\langle x_L^\alpha : L \in \mathcal{L} \rangle$ such that (1)-(3) hold when $\eta = \alpha$. Let $\beta, \gamma < \alpha$ and $L, M, N \in \mathcal{L}$ be arbitrary save $M, N \subseteq L$. Then there exists $P \in \mathcal{L}$ such that $M, N \supseteq P$. Hence, $x_M^\beta, x_N^\gamma \leq x_P^{\max\{\beta,\gamma\}}$. Therefore, $\{x_Q^\delta : \delta < \alpha,\ L \supseteq Q \in \mathcal{L}\}$ is a directed set. Let us denote this set by $X_{\alpha,L}$. Then $|X_{\alpha,L}| < \kappa$ and $X_{\alpha,L} \subseteq L$. For each $Q \in \mathcal{L}$, define x_Q^α as the supremum of $X_{\alpha,Q}$ in Q. Then (1) holds by definition. Also, $x_L^\beta \in X_{\alpha,L}$; hence, $x_L^\beta \leq x_L^\alpha$, which proves (2). Moreover, $x_P^\beta \in X_{\alpha,M}$; hence, $x_P^\beta \leq x_M^\alpha$. Since $x_N^\beta \leq x_P^\beta$, we have $x_N^\beta \leq x_M^\alpha$. Since β and N are arbitrary save $\beta < \alpha$ and $N \subseteq L$, the point x_M^α is an upper bound of $X_{\alpha,L}$. Moreover, $x_M^\alpha \in M \subseteq L$. Therefore, $x_L^\alpha \leq x_M^\alpha$, which proves (3). Thus, by induction, (1)-(3) hold for all $\eta < \kappa$.

Applying (2), we have $x_{L_0}^\alpha \leq x_{L_0}^\beta$ for all $\alpha \leq \beta < \kappa$. But κ cannot be order-embedded in $\langle L_0, \leq \rangle$; hence, there is an $\alpha < \kappa$ such that $x_{L_0}^\alpha = x_{L_0}^{\alpha+1}$. Suppose $M \in \mathcal{L}$ and $M \subseteq L_0$. Then $x_M^\alpha \in X_{\alpha+1,L_0}$; hence, $x_M^\alpha \leq x_{L_0}^{\alpha+1}$. Moreover, (3) implies $x_{L_0}^\alpha \leq x_M^\alpha$. Therefore, $x_{L_0}^\alpha \leq x_M^\alpha \leq x_{L_0}^{\alpha+1} = x_{L_0}^\alpha$; hence, $x_{L_0}^\alpha = x_M^{\alpha+1} \in M$. Since $\langle \mathcal{L}, \supseteq \rangle$ is directed and M is arbitrary save $M \subseteq L_0$, we have $x_{L_0}^\alpha \in \bigcap \mathcal{L}$. Since $x_{L_0}^0$ is an upper bound of E and $x_{L_0}^0 \leq x_{L_0}^\alpha$, the point $x_{L_0}^\alpha$ is an upper bound for E in $\bigcap \mathcal{L}$.

Suppose y is also an upper bound of E in $\bigcap \mathcal{L}$. Then we claim $x_L^\beta \leq y$ for all $L \in \mathcal{L}$ and $\beta < \kappa$. To prove this claim, suppose $\beta < \kappa$ and the claim holds for all

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$\gamma < \beta$. Suppose $\beta = 0$. Let $L \in \mathcal{L}$ be arbitrary. Then y is an upper bound of E in L; hence, $x_L^\beta = x_L^0 \leq y$. Suppose $\beta > 0$ and $\gamma < \beta$ and $L, M \in \mathcal{L}$ and $M \subseteq L$. Then $x_M^\gamma \leq y$ by hypothesis. Hence, y is an upper bound of $X_{\beta,L}$; hence, $x_L^\beta \leq y$. By induction, the claim holds. In particular, $x_{L_0}^\alpha \leq y$; hence, $x_{L_0}^\alpha$ is the least upper bound of E in $\bigcap \mathcal{L}$. □

Corollary 2. *Let $\mathcal{L}$ be a nonempty set and let $\leq$ be a relation on $\bigcup \mathcal{L}$ such that $\langle L, \leq \rangle$ is a complete poset for all $L \in \mathcal{L}$. Suppose that $\langle \mathcal{L}, \supseteq \rangle$ is directed. Then $\langle \bigcap \mathcal{L}, \leq \rangle$ is a complete poset. In particular, $\bigcap \mathcal{L} \neq \emptyset$.*

Proof. Choose an infinite cardinal κ that is greater than both $|\mathcal{L}|$ and $|\bigcup \mathcal{L}|$. Then, by Theorem 1, every empty or directed subset of $\bigcap \mathcal{L}$ has a least upper bound in $\bigcap \mathcal{L}$. □

Theorem 1 has an analogue that is proven in exactly the same manner.

Theorem 3. *Let κ be an infinite cardinal and let $\mathcal{L}$ be a nonempty set of size less than κ such that $\langle \mathcal{L}, \supseteq \rangle$ is directed. Suppose that $\leq$ is a relation on $\bigcup \mathcal{L}$ that partially orders each $L \in \mathcal{L}$. Also suppose that, for each $L \in \mathcal{L}$, every subset of L of size less than κ has a least upper bound with respect to $\langle L, \leq \rangle$. Finally, suppose that there exists an $L_0 \in \mathcal{L}$ such that κ cannot be order-embedded in $\langle L_0, \leq \rangle$. Then every subset of $\bigcap \mathcal{L}$ of size less than κ has a least upper bound with respect to $\langle \bigcap \mathcal{L}, \leq \rangle$. In particular, $\bigcap \mathcal{L} \neq \emptyset$.*

Proof. The proof of Theorem 1 works verbatim, except that we do not require E to be directed. □

Corollary 4. *Let $\mathcal{L}$ be a nonempty set and let $\leq$ be a relation on $\bigcup \mathcal{L}$ such that $\langle L, \leq \rangle$ is a complete lattice for all $L \in \mathcal{L}$. Suppose that $\langle \mathcal{L}, \supseteq \rangle$ is directed. Then $\langle \bigcap \mathcal{L}, \leq \rangle$ is a complete lattice. In particular, $\bigcap \mathcal{L} \neq \emptyset$.*

Proof. Choose an infinite cardinal κ that is greater than both $|\mathcal{L}|$ and $|\bigcup \mathcal{L}|$. Then, by Theorem 3, every subset of $\bigcap \mathcal{L}$ has a least upper bound in $\bigcap \mathcal{L}$. □

Remark 5. Theorems 1 and 3 are stated and proved entirely inside Zermelo-Frankel set theory without choice. The proofs of Corollaries 2 and 4 both use choice.