A COMPACTNESS THEOREM FOR POSETS

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Theorem 1. Let $\kappa$ be an infinite cardinal and let $\mathcal{L}$ be a nonempty set of size less than $\kappa$ such that $\langle \mathcal{L}, \supseteq \rangle$ is directed. Suppose that $\leq$ is a relation on $\bigcup \mathcal{L}$ that partially orders each $L \in \mathcal{L}$. Also suppose that, for each $L \in \mathcal{L}$, every empty or $\leq$-directed subset of $L$ of size less than $\kappa$ has a least upper bound with respect to $\langle L, \leq \rangle$. Finally, suppose that there exists an $L_0 \in \mathcal{L}$ such that $\kappa$ cannot be order-embedded in $\langle L_0, \leq \rangle$. Then every $\leq$-directed subset of $\bigcap \mathcal{L}$ of size less than $\kappa$ has a least upper bound with respect to $\langle \bigcap \mathcal{L}, \leq \rangle$. In particular, $\bigcap \mathcal{L} \neq \emptyset$.

Proof. Let $E \subseteq \bigcap \mathcal{L}$ be such that $|E| < \kappa$ and $E$ is directed. To construct the supremum of $E$ in $\bigcap \mathcal{L}$, let us first recursively define a particular indexed set of points $\langle x^\beta_L : \beta < \kappa, L \in \mathcal{L} \rangle$ and simultaneously prove that, for all $\eta < \kappa$ and $L \in \mathcal{L}$, we have

1. $x^\beta_L \in L$,
2. $x^\beta_L \leq x^\gamma_L$ if $\gamma \leq \beta$, and
3. $x^\beta_L \leq y$ if $L \supseteq M \in \mathcal{L}$.

First, for each $L \in \mathcal{L}$, let $x^0_L$ be the least upper bound of $E$ in $L$. Suppose $L, M \in \mathcal{L}$ and $L \supseteq M$. Then the least upper bound of $E$ in $M$ is an upper bound of $E$ in $L$; hence, $x^0_L \leq x^0_M$. Also, $x^0_L \in L$ is clear. Thus, (1)-(3) hold when $\eta = 0$.

Now suppose $0 < \alpha < \kappa$ and (1)-(3) hold for all $\eta < \alpha$. Let us define $\langle x^\alpha_L : L \in \mathcal{L} \rangle$ such that (1)-(3) hold when $\eta = \alpha$. Let $\beta, \gamma < \alpha$ and $L, M, N \in \mathcal{L}$ be arbitrary save $M, N \subseteq L$. Then there exists $P \in \mathcal{L}$ such that $M, N \supseteq P$. Hence, $x^\beta_M, x^\gamma_N \leq x^\max\{\beta, \gamma\}$. Therefore, $\langle x^\delta_P : \delta < \alpha, L \supseteq P \in \mathcal{L} \rangle$ is a directed set. Let us denote this set by $X_{\alpha,L}$. Then $|X_{\alpha,L}| < \kappa$ and $X_{\alpha,L} \subseteq L$. For each $Q \in \mathcal{L}$, define $x^\alpha_Q$ as the supremum of $X_{\alpha,Q}$ in $Q$. Then (1) holds by definition. Also, $x^\beta_L \in X_{\alpha,L}$; hence, $x^\beta_L \leq x^\alpha_L$, which proves (2). Moreover, $x^\beta_p \in X_{\alpha,M}$; hence, $x^\beta_p \leq x^\alpha_M$. Since $x^\beta_N \leq x^\beta_p$, we have $x^\alpha_N \leq x^\alpha_M$. Since $\beta$ and $N$ are arbitrary save $\beta < \alpha$ and $N \subseteq L$, the point $x^\alpha_M$ is an upper bound of $X_{\alpha,L}$. Moreover, $x^\alpha_M \in M \subseteq L$. Therefore, $x^\alpha_L \leq x^\alpha_M$, which proves (3). Thus, by induction, (1)-(3) hold for all $\eta < \kappa$.

Applying (2), we have $x^\alpha_{L_0} \leq x^\beta_L$, for all $\alpha \leq \beta < \kappa$. But $\kappa$ cannot be order-embedded in $\langle L_0, \leq \rangle$; hence, there is an $\alpha < \kappa$ such that $x^\alpha_{L_0} = x^\alpha_{L_0} + 1$. Suppose $M \in \mathcal{L}$ and $M \subseteq L_0$. Then $x^\alpha_M \in X_{\alpha+1,L_0}$; hence, $x^\alpha_M \leq x^\alpha_{L_0}$. Moreover, (3) implies $x^\alpha_{L_0} \leq x^\alpha_M$. Therefore, $x^\alpha_{L_0} \leq x^\alpha_M \leq x^\alpha_{L_0} + 1 = x^\alpha_{L_0}$; hence, $x^\alpha_{L_0} = x^\alpha_{L_0} + 1 \in M$. Since $\langle \mathcal{L}, \supseteq \rangle$ is directed and $M$ is arbitrary save $M \subseteq L_0$, we have $x^\alpha_{L_0} \in \bigcap \mathcal{L}$. Since $x^\alpha_{L_0}$ is an upper bound of $E$ and $x^\alpha_{L_0} \leq x^\alpha_{L_0}$, the point $x^\alpha_{L_0}$ is an upper bound for $E$ in $\bigcap \mathcal{L}$.

Suppose $y$ is also an upper bound of $E$ in $\bigcap \mathcal{L}$. Then we claim $x^\beta_L \leq y$ for all $L \in \mathcal{L}$ and $\beta < \kappa$. To prove this claim, suppose $\beta < \kappa$ and the claim holds for all
γ < β. Suppose β = 0. Let L ∈ L be arbitrary. Then y is an upper bound of E in L; hence, \( x_L^\beta = x_L^0 \leq y \). Suppose β > 0 and γ < β and L, M ∈ L and M ⊆ L. Then \( x_M^\gamma \leq y \) by hypothesis. Hence, y is an upper bound of \( X_{\beta, L} \); hence, \( x_L^\gamma \leq y \).

By induction, the claim holds. In particular, \( x_L^\alpha \leq y \); hence, \( x_L^\alpha \) is the least upper bound of E in \( \cap L \).

**Corollary 2.** Let L be a nonempty set and let \( \leq \) be a relation on \( \bigcup L \) such that \( \langle L, \leq \rangle \) is a complete poset for all \( L \in L \). Suppose that \( \langle L, \supseteq \rangle \) is directed. Then \( \langle \bigcap L, \leq \rangle \) is a complete poset. In particular, \( \bigcap L \neq \emptyset \).

**Proof.** Choose an infinite cardinal \( \kappa \) that is greater than both \( |L| \) and \( |\bigcup L| \). Then, by Theorem 1, every empty or directed subset of \( \bigcap L \) has a least upper bound in \( \bigcap L \).

Theorem 1 has an analogue that is proven in exactly the same manner.

**Theorem 3.** Let \( \kappa \) be an infinite cardinal and let L be a nonempty set of size less than \( \kappa \) such that \( \langle L, \supseteq \rangle \) is directed. Suppose that \( \leq \) is a relation on \( \bigcup L \) that partially orders each \( L \in L \). Also suppose that, for each \( L \in L \), every subset of L of size less than \( \kappa \) has a least upper bound with respect to \( \langle L, \leq \rangle \). Finally, suppose that there exists an \( L_0 \in L \) such that \( \kappa \) cannot be order-embedded in \( \langle L_0, \leq \rangle \). Then every subset of \( \bigcap L \) of size less than \( \kappa \) has a least upper bound with respect to \( \langle \bigcap L, \leq \rangle \). In particular, \( \bigcap L \neq \emptyset \).

**Proof.** The proof of Theorem 1 works verbatim, except that we do not require E to be directed.

**Corollary 4.** Let L be a nonempty set and let \( \leq \) be a relation on \( \bigcup L \) such that \( \langle L, \leq \rangle \) is a complete lattice for all \( L \in L \). Suppose that \( \langle L, \supseteq \rangle \) is directed. Then \( \langle \bigcap L, \leq \rangle \) is a complete lattice. In particular, \( \bigcap L \neq \emptyset \).

**Proof.** Choose an infinite cardinal \( \kappa \) that is greater than both \( |L| \) and \( |\bigcup L| \). Then, by Theorem 3, every subset of \( \bigcap L \) has a least upper bound in \( \bigcap L \).

**Remark 5.** Theorems 1 and 3 are stated and proved entirely inside Zermelo-Frankel set theory without choice. The proofs of Corollaries 2 and 4 both use choice.