POWER HOMOGENEOUS COMPACTA AND THE ORDER THEORY OF LOCAL BASES

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Abstract. We show that if a power homogeneous compactum X has character κ^+ and density at most κ, then there is a nonempty open $U \subseteq X$ such that every p in U is flat, "flat" meaning that p has a family $\mathcal{F}$ of $\chi(p, X)$-many neighborhoods such that p is not in the interior of the intersection of any infinite subfamiliy of $\mathcal{F}$. The binary notion of a point being flat or not flat is refined by a cardinal function, the local Noetherian type, which is in turn refined by the κ-wide splitting numbers, a new family of cardinal functions we introduce. We show that the flatness of p and the κ-wide splitting numbers of p are invariant with respect to passing from p in X to $\langle p\rangle_{\alpha<\lambda}$ in X^λ, provided that $\lambda < \chi(p, X)$, or, respectively, that $\lambda < \operatorname{cf}\kappa$. The above $<\chi(p, X)$-power-invariance is not generally true for the local Noetherian type of p, as shown by a counterexample where $\chi(p, X)$ is singular.

1. Introduction

Definition 1.1. A space X is *homogeneous* if for any $p, q \in X$ there is a homeomorphism $h\colon X \to X$ such that $h(p) = q$.

There are several known restrictions on the cardinalities of homogeneous compacta. First we mention a classical result, and then we very briefly survey some more recent progress.

Theorem 1.2.

- *Arhangel'skiĭ's Theorem: if X is compact, then $|X| \le 2^{\chi(X)}$.*
- *Čech-Pospišil Theorem: if X is a compactum without isolated points and $\kappa = \min_{p\in X} \chi(p, X)$, then $|X| \ge 2^\kappa$.*
- *Hence, if X is an infinite homogeneous compactum, then $|X| = 2^{\chi(X)}$.*

In constrast to Theorem 1.2, the cardinality of the ordered compactum $\omega_\omega + 1$ is not of the form 2^κ for any κ.

(See Engelking [7], Juhász [8], and Kunen [10] for all undefined terms. Our convention is that $\pi w(\cdot)$, $\chi(\cdot)$, $\pi\chi(\cdot)$, $d(\cdot)$, $c(\cdot)$, and $\mathrm{t}(\cdot)$ respectively denote π-weight, character, π-character, density, cellularity, and tightness of topological spaces.)

Theorem 1.3.

- $|X| \le 2^{\pi\chi(X)c(X)}$ *for every homogeneous T_2 X.* [4]

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- $|X| \leq 2^{\mathrm{t}(X)}$ *for every homogeneous compactum* X. [24]
- $|X| \leq 2^{c(X)}$ *for every* T_5 *homogeneous compactum* X. [13]

In contrast, $|\beta\mathbb{N}| = 2^{2^{\aleph_0}}$ despite $\beta\mathbb{N}$ being compact and having countable π-weight.

Despite the above knowledge (and much more), many important questions about homogeneous compacta remain open. See Van Mill [14] and Kunen [9] to survey these questions. For example, Van Douwen's Problem asks whether there is a homogeneous compactum X with $c(X) > 2^{\aleph_0}$. This question is open in all models of ZFC, and has been open for several decades. (A more general version of this question, also open, asks whether every compactum is a continuous image of a homogeneous compactum.) Milovich [15] connected Van Douwen's Problem with the order theory of local bases through the next theorem. We include a short proof for the reader's convenience.

Definition 1.4.

- A preordered set $\langle P, \leq\rangle$ is *κ-founded* $|\{q \in P : q \leq p\}| < \kappa$ for all $p \in P$.
- A preordered set $\langle P, \leq\rangle$ is *κ^{op}-like* if $|\{q \in P : q \geq p\}| < \kappa$ for all $p \in P$.
- Unless indicated otherwise, families of sets are assumed to be ordered by inclusion.
- For any point p in a space X, the *local Noetherian type* of p in X, or $\chi\mathrm{Nt}(p, X)$, denotes the least infinite cardinal κ for which p has a κ^{op}-like local base in X.
- The local Noetherian type of X, or $\chi\mathrm{Nt}(X)$, denotes
$$\sup_{p\in X} \chi\mathrm{Nt}(p, X).$$
- The Noetherian type of X, or $\mathrm{Nt}(X)$, denotes the least infinite cardinal κ such that X has a κ^{op}-like base.

Malykhin, Peregudov, and Šapirovskiĭ studied the properties $\aleph_1 \geq \mathrm{Nt}(X)$ and $\mathrm{Nt}(X) = \aleph_0$ in the 1970s and 1980s (see, *e.g.*, [11, 18]). Peregudov introduced Noetherian type in 1997 [17]. Bennett and Lutzer rediscovered the property $\mathrm{Nt}(X) = \aleph_0$ in 1998 [3]. In 2008, Milovich introduced local Noetherian type [15].

Lemma 1.5 ([15, Lemma 2.4])**.** *Every preordered set P has a cofinal subset that is $|P|$-founded. Likewise, every family $\mathcal{U}$ of open sets has a dense $|\mathcal{U}|^{\mathrm{op}}$-like subfamily. Hence, $\chi\mathrm{Nt}(p, X) \leq \chi(p, X)$ for all points p in spaces X.*

Lemma 1.6 ([15, Lemma 3.20])**.** *If X is a compactum such that $\chi(X) = \pi\chi(p, X)$ for all $p \in X$, then $\chi\mathrm{Nt}(p, X) = \omega$ for some $p \in X$.*

Theorem 1.7 ([15, Theorem 1.7])**.** *Assuming GCH, if X is a homogeneous compactum, then $\chi\mathrm{Nt}(X) \leq c(X)$.*

Proof. Let X be a homogeneous compactum; we may assume X is infinite. By Theorem 1.3, $|X| \leq 2^{\pi\chi(X)c(X)}$. Since $|X| = 2^{\chi(X)}$ by Theorem 1.2, we have $\chi(X) \leq \pi\chi(X)c(X)$ by GCH. If $\pi\chi(X) = \chi(X)$, then $\chi\mathrm{Nt}(X) = \omega$ by Lemma 1.6. Hence, we may assume $\pi\chi(X) < \chi(X)$; hence, $\chi\mathrm{Nt}(X) \leq \chi(X) \leq c(X)$ by Lemma 1.5. □

Therefore, if, for example, someone proved that there were a model of ZFC + GCH with a homogeneous compactum in which some (equivalently, every) point p had a local base $\mathcal{B}$ such that $\langle \mathcal{B}, \supseteq \rangle$ is isomorphic to $\omega \times \omega_1 \times \omega_2$ with the product order ($\omega \times \omega_2$ would work just as well), then this space would be a consistently existent counterexample for Van Douwen's Problem. Indeed, $\omega \times \omega_1 \times \omega_2$ is not $\aleph_1$-founded and every other local base at p would, by [15, Lemma 2.21], be sufficiently similar (more precisely, Tukey equivalent) to $\omega \times \omega_1 \times \omega_2$ so as to be also not $\aleph_1$-founded. Therefore, Theorem 1.7 implies that the cellularity of such a space would be at least $\aleph_2$.

For example, lexicographically order $X_0 = 2^{\omega}$, $X_1 = 2^{\omega_1}$, and $X_2 = 2^{\omega_2}$, and then form the product $X = \prod_{i<3} X_i$. The space X is compact and every point in X has a local base of type $\omega \times \omega_1 \times \omega_2$. However, X is not homogeneous because there are points $p_0, p_1, p_2 \in X$ such that $\pi\chi(p_i, X) = \aleph_i$ for all $i < 3$. It is not clear whether this obstruction to homogeneity can be bypassed with a more clever example, but Arhangel′skiĭ [1] has shown that if a product of linearly ordered compacta is homogeneous, then every factor is first countable.

Also in [15], a mysterious correlation between the Noetherian types and the cellularities of the known homogeneous compacta is proven. Briefly, every known homogeneous compactum is a continuous image of a product of compacta each with weight at most $2^{\aleph_0}$. Every (known or unknown) homogeneous compactum X that is such a continuous image satisfies $c(X) \leq 2^{\aleph_0}$, $\chi\mathrm{Nt}(X) \leq 2^{\aleph_0}$, and $\mathrm{Nt}(X) \leq \left(2^{\aleph_0}\right)^+$. An important question is whether this correlation has a deep reason, or is merely a coincidence born of ignorance of more exotic homogeneous compacta.

Another curiosity is that although the lexicographic ordering of $2^{\omega\cdot\omega}$ is a homogeneous compactum with cellularity $2^{\aleph_0}$ (see [12]), and the double-arrow space is a homogeneous compactum with Noetherian type $\left(2^{\aleph_0}\right)^+$ (see [15, Example 2.25] or [17]), every known example of a homogeneous compactum X (in any model of ZFC) actually satisfies $\chi\mathrm{Nt}(X) = \omega$ (see [15, Observation 1.4]). In other words, all known homogeneous compacta are *flat.*

Definition 1.8. We say that a point p in a space X is flat if $\chi\mathrm{Nt}(p, X) = \omega$. We say that X is flat if $\chi\mathrm{Nt}(X) = \omega$.

Theorem 2.22 says that p is flat in X if and only if $\langle p \rangle_{i \in I}$ is flat in X^I for all sets I. Moreover, Theorem 2.26 implies that X is flat if and only if X^ω is flat. On the other hand, Example 2.14 shows that for every uncountable cardinal λ, there is a non-flat compactum X such that $\lambda < \mathrm{cf}(\chi(X))$ and X^λ is flat.

To the best of the authors' knowledge, all known *power homogeneous* compacta are also flat.

Definition 1.9 ([5])**.** A space is power homogeneous if some (nonzero) power of it is homogeneous.

There are many inhomogeneous, power homogeneous compacta. For example, Dow and Pearl [6] proved that if X is any first countable, zero dimensional compactum, then X^ω is homogeneous. Nevertheless, homogeneity casts a long shadow over the class of power homogeneous spaces. In particular, Van Douwen's Problem is still open if "homogeneous" is replaced by "power homogeneous." Moreover, many theorems about homogeneous compacta have been shown to hold when "homogeneous" is replaced by "power homogeneous." For example, see [13], as well as the more recent papers cited in the theorem below.

Theorem 1.10.
- $|X| \leq 2^{\pi\chi(X)c(X)}$ *for every power homogeneous Hausdorff* X*.* [4]
- $|X| \leq 2^{\mathrm{t}(X)}$ *for every power homogeneous compactum* X*.* [2]
- $|X| \leq 2^{c(X)}$ *for every* T_5 *homogeneous compactum* X [20]
- $|X| \leq d(X)^{\pi\chi(X)}$ *for every power homogeneous Hausdorff* X*.* [19]

Theorem 1.3's cardinality bound of $2^{\pi\chi(X)c(X)}$ was used in the proof of Theorem 1.7, so it is natural to ask to what extent Theorem 1.7 is true of power homogeneous compacta, which satisfy the same cardinality bound. Specifically, assuming GCH, do all power homogeneous compacta X satisfy $\chi\mathrm{Nt}(X) \leq c(X)$, or at least $\chi\mathrm{Nt}(X) \leq d(X)$? Section 3 presents a partial positive answer to the last question. We show that if $d(X) < \mathrm{cf}\,\chi(X) = \max_{p\in X}\chi(p,X)$, then there is a nonempty open $U \subseteq X$ such that $\chi\mathrm{Nt}(p,X) = \omega$ for all $p \in U$. (Note that $\chi\mathrm{Nt}(X) \leq \chi(X)$.)

Before we can begin Section 3, we must first introduce some more precise order-theoretic cardinal functions, the *κ-wide splitting numbers.*

Definition 1.11.
- Given a space X and $E \subseteq X$, let $\mathrm{int}\,E$ denote the interior of E in X.
- A sequence $\langle U_i\rangle_{i\in I}$ of neighborhoods of a point p in a space X is *λ-splitting* at p if, for all $J \in [I]^\lambda$, we have $p \notin \mathrm{int}\bigcap_{j\in J} U_j$.
- Likewise, a family $\mathcal{F}$ of neighborhoods of p is λ-splitting at p if $p \notin \mathrm{int}\bigcap\mathcal{E}$ for all $\mathcal{E} \in [\mathcal{F}]^\lambda$.
- Given an infinite cardinal κ and a point p in a space X, let the *κ-wide splitting number* of p in X, or $\mathrm{split}_\kappa(p,X)$, denote the least λ such that there exists a λ-splitting sequence $\langle U_\alpha\rangle_{\alpha<\kappa}$ of neighborhoods of p.
- Set $\mathrm{split}_{<\kappa}(p,X) = \sup_{\lambda<\kappa}\mathrm{split}_\lambda(p,X)$. (Declare $\mathrm{split}_{<\omega}(p,X) = \omega$.)
- The κ-wide splitting number of X, or $\mathrm{split}_\kappa(X)$, denotes
$$\sup_{p\in X}\mathrm{split}_\kappa(p,X).$$

Note that if $\kappa \leq \lambda$, then $\operatorname{split}_\kappa(p, X) \leq \operatorname{split}_\lambda(p, X)$. Also, $\kappa^+ \geq \operatorname{split}_\kappa(p, X)$ because a κ-long sequence of open sets is vacuously κ^+-splitting at every point.

The κ-wide splitting numbers are relevant because the local Noetherian type of a point p in a space X is also the $\chi(p, X)$-wide splitting number of p in X:

Proposition 1.12 ([15, Lemma 5.3])**.** *If $\kappa = \chi(p, X)$ and p does not have a finite local base, then $\chi\mathrm{Nt}(p, X) = \operatorname{split}_\kappa(p, X)$.*

Thus, if $\kappa \leq \chi(p, X)$, then $\operatorname{split}_\kappa(p, X) \leq \chi\mathrm{Nt}(p, X) \leq \chi(p, X)$.

Section 3 requires some basic knowledge of how the κ-wide splitting numbers are affected by passing from a space X to a power of X. This question is investigated in depth in in Section 2. An oversimplified answer is that the κ-wide splitting number does not change as we pass from smaller powers of X to higher powers of X, except at X^κ, and possibly at $X^{\operatorname{cf}\kappa}$. In fact, the κ-wide splitting number always collapses to ω at X^κ. If κ is singular, then the κ-wide splitting number might also make a change of form λ^+ to λ at $X^{\operatorname{cf}\kappa}$.

The least easy (and most novel) results of Section 2 involve limit cardinals. From a purely technical point of view, three examples are the most interesting results of this section.

- Example 2.28 gives a (simultaneous) instance of
$$\aleph_1 \leq \tau = \operatorname{cf}(\chi(p, X)) < \chi(p, X)$$
and $\chi\mathrm{Nt}(p, X) > \chi\mathrm{Nt}(\langle p\rangle_{\alpha<\tau}, X^\tau)$ (assuming only ZFC). Theorem 2.26 shows that the condition $\aleph_1 \leq \tau$ is necessary.
- Example 2.29 shows that as λ increases, the λ-wide splitting number can jump from ω to κ at $\lambda = \kappa$ if κ is strongly inaccessible; Question 2.30 asks if this is possible for merely weakly inaccessible κ.
- Example 2.11 gives an instance of
$$\chi\mathrm{Nt}(p, X^2) < \min_{i<2} \chi\mathrm{Nt}(p(i), X)$$
(assuming only ZFC). PFA implies that any instance of this inequality must satisfy $\chi(p, X^2) \geq \aleph_2$, but CH implies there is an instance satisfying $\chi(p, X^2) = \aleph_1$. ($\chi(p, X^2) \geq \aleph_1$ is trivially necessary.)

2. λ-splitting families and products

Lemma 2.1. *Suppose $f\colon X \to Y$ and $p \in X$ and f is continuous at p and open at p. We then have $\operatorname{split}_\kappa(p, X) \leq \operatorname{split}_\kappa(f(p), Y)$ for all κ.*

Proof. Set $\lambda = \operatorname{split}_\kappa(f(p), Y)$ and let $\langle V_\alpha\rangle_{\alpha<\kappa}$ be a λ-splitting sequence of neighborhoods of $f(p)$. For each $\alpha < \kappa$, let $U_\alpha = f^{-1}[V_\alpha]$. Suppose $I \in [\kappa]^\lambda$. We then have $f(p) \notin \operatorname{int}\bigcap_{\alpha\in I} V_\alpha$. If $p \in \operatorname{int}\bigcap_{\alpha\in I} U_\alpha$, then $f(p) \in \operatorname{int} f\left[\bigcap_{\alpha\in I} U_\alpha\right] \subseteq \operatorname{int}\bigcap_{\alpha\in I} V_\alpha$, which is absurd. Thus, $p \notin \operatorname{int}\bigcap_{\alpha\in I} U_\alpha$, so $\operatorname{split}_\kappa(p, X) \leq \lambda$. $\square$

Since coordinate projections are continuous and open everywhere, we will use Lemma 2.1 many times in this section. We only use the full strength of the lemma in Section 3.

Lemma 2.1 is a modification of a theorem of [16] which states that if f, p, X, Y are as in the lemma, $\mathcal{A}$ is a local base at p, and $\mathcal{B}$ is a local base at $f(p)$, then there is a Tukey map from $\langle \mathcal{B}, \supseteq \rangle$ to $\langle \mathcal{A}, \supseteq \rangle$, where a map between preorders is *Tukey* [23] if every subset of the domain without an upper bound in the domain is mapped to a set without an upper bound in the codomain. (A particularly useful special case occurs when f is the identity map on X, that is, when $\mathcal{A}$ and $\mathcal{B}$ are local bases at the same point.) [15, Lemma 5.8] says that a point p in a space X is flat if and only if there is a Tukey map from $\langle [\chi(p, X)]^{<\omega}, \subseteq \rangle$ to $\langle \mathcal{A}, \supseteq \rangle$ for some (equivalently, every) local base $\mathcal{A}$ at p. Moreover, it is a standard (easy) result that if κ is an infinite cardinal and P is a directed set, then there is a Tukey map from $[\kappa]^{<\omega}$ to P if and only if P has a subset S of size κ such that no infinite subset of S is bounded. Hence, $\mathrm{split}_\kappa(p, X) = \omega$ if and only if there is a Tukey map from $[\kappa]^{<\omega}$ to $\langle \mathcal{A}, \supseteq \rangle$ for some (equivalently, every) local base $\mathcal{A}$ at p. We will use Tukey maps in Example 2.11.

Lemma 2.2. *If* $\chi(p, X) < \mathrm{cf}\,\kappa$ *or* p *has a finite local base, then* $\mathrm{split}_\kappa(p, X) = \kappa^+$. *If* p *has no finite local base and* $\mathrm{cf}\,\kappa \leq \chi(p, X) < \kappa$*, then* $\mathrm{split}_\kappa(p, X) \geq \kappa$ *and* $\mathrm{split}_\kappa(p, X) = \kappa$ *if and only if* $\mathrm{split}_{\mathrm{cf}\,\kappa}(p, X) \leq \mathrm{cf}\,\kappa$.

Proof. Let $\langle U_\beta \rangle_{\beta<\kappa}$ be a sequence of neighborhoods of p. If p has a local base $\mathcal{F}$ such that $|\mathcal{F}| < \mathrm{cf}\,\kappa$, then some $H \in \mathcal{F}$ is contained in U_α for κ-many α. Therefore, we may assume that p does not have a finite local base and that $\mathrm{cf}\,\kappa \leq \chi(p, X) < \kappa$. Let $\langle \lambda_\alpha \rangle_{\alpha<\mathrm{cf}\,\kappa}$ be an increasing sequence of regular cardinals cofinal in κ such that $\chi(p, X) < \lambda_0$. For each $\alpha < \mathrm{cf}\,\kappa$, choose $I_\alpha \in [\lambda_\alpha]^{\lambda_\alpha}$ such that $V_\alpha = \mathrm{int} \bigcap_{\beta \in I_\alpha} U_\beta$ is nonempty. The sequence $\langle I_\alpha \rangle_{\alpha<\mathrm{cf}\,\kappa}$ witnesses that $\mathrm{split}_\kappa(p, X) \geq \kappa$. Moreover, if $\langle U_\beta \rangle_{\beta<\kappa}$ is κ-splitting, then $\langle V_\alpha \rangle_{\alpha<\mathrm{cf}\,\kappa}$ is $(\mathrm{cf}\,\kappa)$-splitting. Conversely, if $\langle W_\alpha \rangle_{\alpha<\mathrm{cf}\,\kappa}$ is $(\mathrm{cf}\,\kappa)$-splitting and $\langle \kappa_\alpha \rangle_{\alpha<\mathrm{cf}\,\kappa}$ is a continuously increasing sequence cofinal in κ, then $\bigcup_{\alpha<\mathrm{cf}\,\kappa} \langle W_\alpha : \beta \in [\kappa_\alpha, \kappa_{\alpha+1}) \rangle$ is κ-splitting. □

Definition 2.3.

- Given a sequence of spaces $\langle X_i \rangle_{i \in I}$ and an infinite cardinal κ, let $\prod_{i \in I}^{(\kappa)} X_i$ denote the set $\prod_{i \in I} X_i$ with the topology generated by the sets of the form $\prod_{i \in I} U_i$ where each U_i is open in X_i and $|\{i \in I : U_i \neq X_i\}| < \kappa$.
- A point p in a space X is a P_κ-point if κ is an infinite cardinal and every intersection of fewer than κ-many neighborhoods of p is itself a neighborhood of p.

Remark.

- $\prod_{i \in I}^{(\omega)} X_i$ is the product space $\prod_{i \in I} X_i$.
- $\prod_{i \in I}^{(\kappa)} X_i$ is the box product space $\square_{i \in I} X_i$ when $\kappa > |I|$.
- $P_{\aleph_1}$-points are also called P-points.
- Every isolated point is a P_κ-point for all κ.

Definition 2.4. Given a subset E of a product $\prod_{i\in I} X_i$ and a subset J of I, we say that E is supported on J, or $\operatorname{supp}(E) \subseteq J$, if $E = (\pi^I_J)^{-1}\left[\pi^I_J[E]\right]$. If there is a least set J for which E is supported on J, then we may write $\operatorname{supp}(E) = J$.

Remark. We always have that $\operatorname{supp}(E) \subseteq A$ and $\operatorname{supp}(E) \subseteq B$ together imply $\operatorname{supp}(E) \subseteq A \cap B$. If a subset E of a product space is itself a product or is open, closed, or finitely supported, then there exists J such that $\operatorname{supp}(E) = J$, so we may unambiguously speak of $\operatorname{supp}(E)$.

Lemma 2.5. *Suppose that κ and μ are infinite cardinals and $\operatorname{cf}\kappa \neq \operatorname{cf}\mu$. If $\xi_\alpha < \mu$ for all $\alpha < \kappa$, then there exists $I \in [\kappa]^\kappa$ such that $\sup_{\alpha\in I} \xi_\alpha < \mu$.*

Proof. Let $\langle \mu_\beta\rangle_{\beta<\operatorname{cf}\mu}$ be a continuously increasing sequence cofinal in μ. Define $f\colon \kappa \to \operatorname{cf}\mu$ by $\xi_\alpha \in [\mu_{f(\alpha)}, \mu_{f(\alpha)+1})$. It suffices to prove that $|f[I]| < \operatorname{cf}\mu$ for some $I \in [\kappa]^\kappa$. If $\operatorname{cf}\kappa > \operatorname{cf}\mu$, then f is constant of a set of size κ. If $\kappa < \operatorname{cf}\mu$, then $|f[\kappa]| < \operatorname{cf}\mu$. Therefore, we may assume $\operatorname{cf}\kappa \leq \operatorname{cf}\mu < \kappa$. Let $\langle \kappa_\gamma\rangle_{\gamma<\operatorname{cf}\kappa}$ be an increasing sequence of regular cardinals cofinal in κ, with $\kappa_0 > \operatorname{cf}\mu$. For each $\gamma < \operatorname{cf}\kappa$, choose $I_\gamma \in [\kappa_\gamma]^{\kappa_\gamma}$ such that f is constant on I_γ. Set $I = \bigcup_{\gamma<\operatorname{cf}\kappa} I_\gamma$, which has size κ. We then have $|f[I]| \leq \operatorname{cf}\kappa < \operatorname{cf}\mu$ as desired. □

Theorem 2.6. *Let κ, λ, μ be infinite cardinals with $\mu \leq \lambda^+$, let $p \in X = \prod^{(\mu)}_{\alpha<\lambda} X_\alpha$, let each $p(\alpha)$ have a neighborhood in X_α other than X_α, and let $p(\alpha)$ be a P_μ-point in X_α, for all $\alpha < \lambda$. We then have:*

$$\kappa < \operatorname{cf}\mu \Rightarrow \operatorname{split}_\kappa(p, X) = \kappa^+; \tag{2.1}$$

$$\operatorname{cf}\kappa = \operatorname{cf}\mu \leq \kappa < \mu \Rightarrow \operatorname{split}_\kappa(p, X) = \kappa; \tag{2.2}$$

$$\operatorname{cf}\kappa \neq \operatorname{cf}\mu < \kappa < \mu \Rightarrow \operatorname{split}_\kappa(p, X) = \kappa^+; \tag{2.3}$$

$$\mu \leq \kappa \leq \lambda \Rightarrow \operatorname{split}_\kappa(p, X) = \mu; \tag{2.4}$$

$$\lambda^+ \leq \kappa \leq \chi(p, X) \Rightarrow \mu \leq \operatorname{split}_\kappa(p, X) \leq \chi(p, X); \tag{2.5}$$

$$\chi(p, X) < \operatorname{cf}\kappa \Rightarrow \operatorname{split}_\kappa(p, X) = \kappa^+. \tag{2.6}$$

$$\operatorname{split}_{\operatorname{cf}\kappa}(p, X) \leq \operatorname{cf}(\kappa) \leq \chi(p, X) < \kappa \Rightarrow \operatorname{split}_\kappa(p, X) = \kappa. \tag{2.7}$$

$$\operatorname{split}_{\operatorname{cf}\kappa}(p, X) > \operatorname{cf}(\kappa) \leq \chi(p, X) < \kappa \Rightarrow \operatorname{split}_\kappa(p, X) = \kappa^+. \tag{2.8}$$

Proof. To prove (2.1), simply observe that every intersection of κ-many neighborhoods of p is itself a neighborhood of p, for all $\kappa < \operatorname{cf}\mu$. This observation also implies that if $\kappa \geq \operatorname{cf}\mu$, $\operatorname{split}_\kappa(p, X) \geq \operatorname{cf}\mu$.

To prove (2.3), let $\langle B_\alpha\rangle_{\alpha<\kappa}$ be a sequence of neighborhoods of p. Let us show that $\langle B_\alpha\rangle_{\alpha<\kappa}$ is not κ-splitting at p. We may assume that each B_α is an open box. By Lemma 2.5, there exist $I \in [\kappa]^\kappa$ and $\nu < \mu$ such that $|\operatorname{supp}(B_\alpha)| \leq \nu$ for all $\alpha \in I$. The box $\bigcap_{\alpha\in I} B_\alpha$ has support of size less than μ; hence, $\langle B_\alpha\rangle_{\alpha<\kappa}$ is not κ-splitting at p; hence, $\operatorname{split}_\kappa(p, X) = \kappa^+$.

To prove (2.2), first consider the case $\kappa = \operatorname{cf}\mu$. We have $\operatorname{split}_{\operatorname{cf}\mu}(p, X) \geq \operatorname{cf}\mu$ from (2.1). To see that $\operatorname{split}_{\operatorname{cf}\mu}(p, X) \leq \operatorname{cf}\mu$, observe that if $\langle A_\alpha\rangle_{\alpha<\operatorname{cf}\mu}$ is a sequence of open boxes each containing p, and we have

$$\sup_{\alpha<\operatorname{cf}\mu} |\operatorname{supp}(A_\alpha)| = \mu,$$

then $\langle A_\alpha \rangle_{\alpha<\operatorname{cf}\mu}$ is $(\operatorname{cf}\mu)$-splitting at p.

Now suppose that $\operatorname{cf}\kappa = \operatorname{cf}\mu < \kappa < \mu$. The cardinal κ must be a limit cardinal, so $\operatorname{split}_\kappa(p, X) \geq \kappa$ by (2.3). Let $\langle \kappa_\alpha \rangle_{\alpha<\operatorname{cf}\kappa}$ be continuously increasing and cofinal in κ; let $\langle \mu_\alpha \rangle_{\alpha<\operatorname{cf}\kappa}$ be increasing and cofinal in μ. Since μ also must be a limit cardinal, each μ_α is less than λ. Hence, we may choose a sequence $\langle C_\beta \rangle_{\beta<\kappa}$ of neighborhoods of p such that, for all $\alpha < \operatorname{cf}\kappa$ and $\beta \in [\kappa_\alpha, \kappa_{\alpha+1})$, C_β is a box with support of size μ_α. For all $J \in [\kappa]^\kappa$, we have $|\{\alpha : J \cap [\kappa_\alpha, \kappa_{\alpha+1}) \neq \varnothing\}| = \operatorname{cf}\kappa$; hence, the support of $\bigcap_{\beta\in J} C_\beta$ has size μ. Therefore, $\langle C_\beta \rangle_{\beta<\kappa}$ is κ-splitting. This completes the proof of (2.2).

Let us prove (2.4). Suppose $\mu \leq \kappa \leq \lambda$. By (2.1) for regular μ and (2.3) for singular μ, $\operatorname{split}_\kappa(p, X) \geq \mu$. Moreover, using an idea of Malykhin [11], we can choose a family of κ-many neighborhoods of p with pairwise disjoint supports; any such family is μ-splitting at p.

Finally, (2.5) follows from (2.1) for regular μ and from (2.3) for singular μ. (2.6), (2.7), and (2.8) are just instances of Lemma 2.2. $\square$

Remark. Concerning (2.5) of Theorem 2.6, Kojman and Milovich have independently shown in unpublished work that if $X = \prod^{(\aleph_1)}_{\alpha<\aleph_\omega} 2$, then GCH+$\square_{\aleph_\omega}$ implies $\chi\mathrm{Nt}(X) = \mathrm{Nt}(X) = \aleph_1$. Soukup has shown that GCH and Chang's Conjecture at $\aleph_\omega$ together imply $\chi\mathrm{Nt}(X) = \mathrm{Nt}(X) = \aleph_2$. [21]

Corollary 2.7 ([15, Theorem 2.33]). *If p and X are as in the above theorem and $\mu = \omega$ (i.e., X is a product space), and $\lambda \geq \chi(p, X)$, then $\chi\mathrm{Nt}(p, X) = \omega$. Hence, if $\lambda \geq \chi(X)$, then $\chi\mathrm{Nt}(X) = \omega$. In particular, $\chi\mathrm{Nt}(Y^{\chi(Y)}) = \omega$ for all spaces Y.*

Thus, large powers are flat, by which we mean that sufficiently large powers of a space X collapse the local Noetherian type (and the κ-wide splitting number for any fixed κ) to ω. We will find more complex behavior at smaller powers of X.

Definition 2.8.

- Given I and p, let $\Delta_I(p)$ denote the constant function $\langle p \rangle_{i\in I}$.
- Let $\operatorname{split}^I_\kappa(p, X)$ denote $\operatorname{split}_\kappa(\Delta_I(p), X^I)$.
- Let $\chi\mathrm{Nt}^I(p, X)$ denote $\chi\mathrm{Nt}(\Delta_I(p), X^I)$.
- All our statements implicitly exclude the case of the product space with no factors, *e.g.*, X^0.

Lemma 2.9. *Suppose p is a point in a space X and $n < \omega$. We then have $\operatorname{split}^n_\kappa(p, X) = \operatorname{split}_\kappa(p, X)$ for all κ.*

Proof. By Lemma 2.1, it suffices to show that $\operatorname{split}^n_\kappa(p, X) \geq \operatorname{split}_\kappa(p, X)$. Set $\lambda = \operatorname{split}^n_\kappa(p, X)$ and let $\langle V_\alpha \rangle_{\alpha<\kappa}$ be a λ-splitting sequence of neighborhoods of $\Delta_n(p)$. Shrinking each V_α to a smaller neighborhood of $\Delta_n(p)$ cannot harm the λ-splitting property, so we may assume that each V_α is a finite product $\prod_{i<n} V_{\alpha,i}$ of open sets. Set $U_\alpha = \bigcap_{i<n} V_{\alpha,i}$ for all α. Suppose $I \in [\kappa]^\lambda$. We then have $\Delta_n(p) \notin \operatorname{int}\bigcap_{\alpha\in I} V_\alpha$. If $p \in \operatorname{int}\bigcap_{\alpha\in I} U_\alpha$, then $\Delta_n(p) \in \left(\operatorname{int}\bigcap_{\alpha\in I} U_\alpha\right)^n \subseteq \operatorname{int}\bigcap_{\alpha\in I} V_\alpha$, which is absurd. Thus, $p \notin \operatorname{int}\bigcap_{\alpha\in I} U_\alpha$, so $\operatorname{split}_\kappa(p, X) \leq \lambda$. $\square$

Theorem 2.10. *Suppose p is a point in a space X and $n < \omega$. We then have $\chi\mathrm{Nt}^n(p, X) = \chi\mathrm{Nt}(p, X)$. Hence, $\chi\mathrm{Nt}(X^n) = \chi\mathrm{Nt}(X)$.*

Proof. The first half of the theorem immediately follows from Lemma 2.9 with $\kappa = \chi(p, X) = \chi(\Delta_n(p), X^n)$. Moreover, the first half immediately implies that $\chi\mathrm{Nt}(X) \le \chi\mathrm{Nt}(X^n)$. To see that $\chi\mathrm{Nt}(X) \ge \chi\mathrm{Nt}(X^n)$, observe that by Lemma 2.1, we have

$$\begin{aligned}\forall q \in X^n \quad \chi\mathrm{Nt}(q, X^n) &= \mathrm{split}_{\chi(q,X^n)}(q, X^n)\\ &\le \mathrm{split}_{\chi(q(i),X)}(q(i), X) = \chi\mathrm{Nt}(q(i), X)\end{aligned}$$

where i is chosen such that $\chi(q, X^n) = \chi(q(i), X)$. □

The following example shows that the natural generalization of Theorem 2.10 to arbitrary points in X^n, namely

$$\chi\mathrm{Nt}(p, X^n) = \min_{i<n} \chi\mathrm{Nt}(p(i), X),$$

fails in general.

Example 2.11. Let κ be a regular uncountable cardinal satisfying $\kappa^{\aleph_0} = \kappa$. For example, κ could be $\left(2^{\aleph_0}\right)^+$ (in any model of ZFC), $2^{\aleph_0}$ if $2^{\aleph_0}$ is regular, or $\aleph_1$ if CH holds. Let $S_0, S_1 \subseteq \kappa$ be stationary with nonstationary intersection. For each $i < 2$, let D_i denote the set of countable subsets of S_i that are compact as subspaces of κ with the order topology. Todorčević [22] has shown that there are no Tukey maps from $\langle [\kappa]^{<\omega}, \subseteq\rangle$ to any $\langle D_i, \subseteq\rangle$, but there is a Tukey map from $\langle [\kappa]^{<\omega}, \subseteq\rangle$ to $\langle D_0 \times D_1, \subseteq\rangle$. For each $i < 2$, let X_i be the set $\kappa \cup \{\infty\}$ topologized such that κ is a discrete subspace and $\mathcal{A}_i = \{X_i \setminus E : E \in D_i\}$ is a local base at ∞. Let X be the topological sum $\bigcup_{i<2}(\{i\} \times X_i)$. Define $p \in X^2$ by $p(i) = \langle i, \infty\rangle$ for all $i < 2$. Since $\kappa^{\aleph_0} = \kappa$, $\chi(p(i), X) = \kappa$ for each $i < 2$. Therefore, there are no Tukey maps from $\langle [\chi(p(i), X)]^{<\omega}, \subseteq\rangle$ to $\langle \mathcal{A}_i, \supseteq\rangle$ for any $i < 2$, but there is a Tukey map from $\langle [\chi(p, X^2)]^{<\omega}, \subseteq\rangle$ to $\langle \mathcal{A}_0 \times \mathcal{A}_1, \supseteq\rangle$. Hence, $\chi\mathrm{Nt}(p(i), X) > \omega$ for all $i < 2$, yet $\chi\mathrm{Nt}(p, X^2) = \omega$. Moreover, for each $i < 2$, $\chi\mathrm{Nt}(p(i), X) = \aleph_1$ because $\langle \{i\} \times (X_i \setminus \{\alpha\})\rangle_{\alpha<\kappa}$ is $\aleph_1$-splitting at $p(i)$.

Remark. If, for each $i < 2$, we replace each isolated point in X_i with an open subspace homeomorphic to 2^κ, then $\chi\mathrm{Nt}(X_0) = \chi\mathrm{Nt}(X_1) = \aleph_1$ and $\chi\mathrm{Nt}(X_0 \times X_1) = \aleph_0$.

Remark. PFA is relevant to the above example, for it implies that if P_0 and P_1 are directed sets of cofinality at most $\aleph_1$ and there is a Tukey map from $\langle [\aleph_1]^{<\omega}, \subseteq\rangle$ to $P_0 \times P_1$, then there is also a Tukey map from $\langle [\aleph_1]^{<\omega}, \subseteq\rangle$ to some P_i [22]. Hence, PFA (which contradicts CH) implies that if $\chi(p, X^n) \le \aleph_1$, then $\chi\mathrm{Nt}(p, X^n) = \min_{i<n} \chi\mathrm{Nt}(p(i), X)$.

Lemma 2.12. *Suppose p is a point in a space X, κ is an infinite cardinal, and $\gamma < \operatorname{cf}\kappa$. We then have*

$$\mathrm{split}_\kappa^\gamma(p, X) = \mathrm{split}_\kappa(p, X).$$

Proof. By Lemma 2.1, it suffices to show that $\mathrm{split}_\kappa^\gamma(p,X) \geq \mathrm{split}_\kappa(p,X)$. Set $\lambda = \mathrm{split}_\kappa^\gamma(p,X)$ and let $\langle V_\alpha\rangle_{\alpha<\kappa}$ be a λ-splitting sequence of neighborhoods of $\Delta_\gamma(p)$. We may assume each V_α has finite support and therefore choose $\sigma_\alpha \in \mathrm{Fn}(\gamma, \{U \subseteq X : U \text{ open}\})$ such that $V_\alpha = \bigcap_{\langle\beta,U\rangle\in\sigma_\alpha} \pi_\beta^{-1}U$. Since $|[\gamma]^{<\omega}| < \mathrm{cf}\,\kappa$, we may assume there is some $s \in [\gamma]^{<\omega}$ such that $\mathrm{dom}\,\sigma_\alpha = s$ for all $\alpha < \kappa$. But then $\langle \pi_s^\gamma[V_\alpha]\rangle_{\alpha<\kappa}$ is λ-splitting at $\Delta_s(p)$ in X^s. Thus, $\mathrm{split}_\kappa^\gamma(p,X) \geq \mathrm{split}_\kappa^s(p,X)$. Apply Lemma 2.9. □

The following corollary is immediate.

Corollary 2.13. *If $\gamma < \mathrm{cf}(\chi(p,X))$, then $\chi\mathrm{Nt}(p,X) = \chi\mathrm{Nt}^\gamma(p,X)$.*

The next example shows that the above corollary is not generally true if we replace the local quantities $\chi(p,X)$, $\chi\mathrm{Nt}(p,X)$, and $\chi\mathrm{Nt}^\gamma(p,X)$ with their global counterparts $\chi(X)$, $\chi\mathrm{Nt}(X)$, and $\chi\mathrm{Nt}(X^\gamma)$.

Example 2.14. For every uncountable cardinal λ, there is a compactum X such that $\lambda < \mathrm{cf}(\chi(X))$ and $\chi\mathrm{Nt}(X^\lambda) = \omega < \lambda = \chi\mathrm{Nt}(X)$. Choose μ such that $\mathrm{cf}\,\mu > \lambda$ and set $X = (\lambda+1)\oplus 2^\mu$, making $\chi(X) = \mu$. By Corollary 2.7, $\chi\mathrm{Nt}(2^\mu) = \omega$, so $\chi\mathrm{Nt}(X) = \chi\mathrm{Nt}(\lambda+1) = \lambda$ (because every regular $\kappa \in \lambda+1$ is a P_κ-point). Set $Y = X^\lambda$. If $p \in Y$ and $p(\alpha) \in 2^\mu$ for some α, then we have

$$\chi\mathrm{Nt}(p,Y) = \mathrm{split}_\mu(p,Y) \leq \mathrm{split}_\mu(p(\alpha),X) = \mathrm{split}_\mu(p(\alpha),2^\mu) = \omega$$

by Lemma 2.1 and Corollary 2.7. If $p \in Y$ and $p(\alpha) \in \lambda+1$ for all $\alpha < \lambda$, then we have

$$\chi\mathrm{Nt}(p,Y) = \mathrm{split}_\lambda(p,Y) = \omega$$

by Corollary 2.7.

Lemma 2.15. *Let p be a point in a space X and let κ,λ be infinite cardinals. If $\mathrm{split}_{<\kappa}(p,X) \leq \lambda$ and $\mathrm{split}_{\mathrm{cf}\,\kappa}(p,X) \leq \mathrm{cf}\,\lambda$, then $\mathrm{split}_\kappa(p,X) \leq \lambda$.*

Proof. Let $\langle U_\alpha : \alpha < \mathrm{cf}\,\kappa\rangle$ be $(\mathrm{cf}\,\lambda)$-splitting at p. Let $\langle\kappa_\alpha\rangle_{\alpha<\mathrm{cf}\,\kappa}$ be a continuously increasing sequence cofinal in κ. For each $\alpha < \mathrm{cf}\,\kappa$, let $\langle V_\beta : \kappa_\alpha \leq \beta < \kappa_{\alpha+1}\rangle$ be λ-splitting at p. For each $\alpha < \mathrm{cf}\,\kappa$ and $\beta \in [\kappa_\alpha,\kappa_{\alpha+1})$, set $W_\beta = U_\alpha \cap V_\beta$. It suffices to show that $\langle W_\beta\rangle_{\beta<\kappa}$ is λ-splitting at p.

Let $I \in [\kappa]^\lambda$. Set $J = \{\alpha < \mathrm{cf}\,\kappa : I \cap [\kappa_\alpha,\kappa_{\alpha+1}) \neq \varnothing\}$. If $|J| \geq \mathrm{cf}\,\lambda$, then $\mathrm{int}\bigcap_{\beta\in I} W_\beta \subseteq \mathrm{int}\bigcap_{\alpha\in J} U_\alpha = \varnothing$. If $|J| < \mathrm{cf}\,\lambda$, then we may choose α such that $|I\cap[\kappa_\alpha,\kappa_{\alpha+1})| = \lambda$. In this case, $\mathrm{int}\bigcap_{\beta\in I} W_\beta \subseteq \mathrm{int}\bigcap_{\beta\in I\cap[\kappa_\alpha,\kappa_{\alpha+1})} W_\beta = \varnothing$. Thus, $\langle W_\beta\rangle_{\beta<\kappa}$ is λ-splitting at p. □

Theorem 2.16. *Let p be a point in a space X, let p have a neighborhood other than X, and let κ and λ be infinite cardinals. We then have*

$$\mathrm{split}_\kappa^\lambda(p,X) = \begin{cases} \mathrm{split}_\kappa(p,X): & \lambda < \mathrm{cf}\,\kappa \\ \mathrm{split}_{<\kappa}(p,X): & \mathrm{cf}\,\kappa \leq \lambda < \kappa \\ \omega: & \kappa \leq \lambda \end{cases}.$$

Proof. The first case of the theorem is just Lemma 2.12. The third case is an instance of Theorem 2.6 with $\mu = \omega$. Consider the second case. Suppose $\lambda <$

$\operatorname{cf}\mu = \mu < \kappa$. By Lemma 2.12, $\operatorname{split}_\mu(p, X) = \operatorname{split}^\lambda_\mu(p, X) \leq \operatorname{split}^\lambda_\kappa(p, X)$. Hence, $\operatorname{split}_{<\kappa}(p, X) \leq \operatorname{split}^\lambda_\kappa(p, X)$. Hence, it suffices to show that

$$\operatorname{split}^\lambda_\kappa(p, X) \leq \operatorname{split}_{<\kappa}(p, X).$$

Since $\operatorname{cf}\kappa \leq \lambda$, we have $\operatorname{split}^\lambda_{\operatorname{cf}\kappa}(p, X) = \omega$ by the third case. Hence,

$$\operatorname{split}^\lambda_{\operatorname{cf}\kappa}(p, X) \leq \operatorname{cf}(\operatorname{split}_{<\kappa}(p, X)).$$

By Lemma 2.1, we also have $\operatorname{split}^\lambda_{<\kappa}(p, X) \leq \operatorname{split}_{<\kappa}(p, X)$. Hence,

$$\operatorname{split}^\lambda_\kappa(p, X) \leq \operatorname{split}_{<\kappa}(p, X)$$

by Lemma 2.15. □

Example 2.17. If $p = \omega_{\omega+1}$ and $X = \omega_{\omega+1} + 1$ (with the order topology), then p is a $P_{\aleph_{\omega+1}}$-point in X, so $\operatorname{split}^\omega_{\aleph_\omega}(p, X) = \operatorname{split}_{<\aleph_\omega}(p, X) = \aleph_\omega$ and $\operatorname{split}_{\aleph_\omega}(p, X) = \aleph_{\omega+1}$.

Given the above theorem, it is natural to investigate the relationship between $\operatorname{split}_\kappa(p, X)$ and $\operatorname{split}_{<\kappa}(p, X)$.

Theorem 2.18. *If p be a point in a space X and κ is a singular cardinal, then*

$$\operatorname{split}_\kappa(p, X) \in \{\operatorname{split}_{<\kappa}(p, X), \operatorname{split}_{<\kappa}(p, X)^+\}.$$

Proof. Trivially, $\operatorname{split}_{\operatorname{cf}\kappa}(p, X) \leq \operatorname{split}_{<\kappa}(p, X)$. Hence, by Lemma 2.15 with $\lambda = \operatorname{split}_{<\kappa}(p, X)^+$, we have $\operatorname{split}_\kappa(p, X) \leq \operatorname{split}_{<\kappa}(p, X)^+$. □

The following corollary is immediate.

Corollary 2.19. *If $\operatorname{cf}(\chi(p, X)) \leq \gamma < \chi(p, X)$, then*

$$\chi\mathrm{Nt}(p, X) \in \{\chi\mathrm{Nt}^\gamma(p, X), \chi\mathrm{Nt}^\gamma(p, X)^+\}.$$

Lemma 2.20. *If $\kappa = \chi(p, X)$ and p has no finite local base, then*

$$\operatorname{split}_{\operatorname{cf}\kappa}(p, X) \leq \operatorname{cf}\kappa.$$

Proof. Let $\bigcup_{\alpha<\operatorname{cf}\kappa} \mathcal{A}_\alpha$ be a local base at p such that $|\mathcal{A}_\alpha| < \kappa$ for all $\alpha < \operatorname{cf}\kappa$. For each $\alpha < \operatorname{cf}\kappa$, set $\mathcal{B}_\alpha = \left\{U \in \mathcal{A}_\alpha : \forall V \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta \;\; V \not\subseteq U\right\}$. Set $I = \{\alpha < \operatorname{cf}\kappa : \mathcal{B}_\alpha \neq \varnothing\}$. Set $\mathcal{B} = \bigcup_{\alpha\in I} \mathcal{B}_\alpha$, which is a local base at p. We then have $|\mathcal{B}| = \kappa$, so $|I| = \operatorname{cf}\kappa$. For each $\alpha \in I$, choose $U_\alpha \in \mathcal{B}_\alpha$. It suffices to show that $\langle U_\alpha\rangle_{\alpha\in I}$ is $(\operatorname{cf}\kappa)$-splitting. Seeking a contradiction, suppose $J \in [I]^{\operatorname{cf}\kappa}$ and $p \in \operatorname{int}\bigcap_{\alpha\in J} U_\alpha$. Choose $V \in \mathcal{B}$ such that $V \subseteq \bigcap_{\alpha\in J} U_\alpha$. Choose $\beta < \operatorname{cf}\kappa$ such that $V \in \mathcal{B}_\beta$. Choose $\alpha \in J$ such that $\beta < \alpha$. We then have $\mathcal{A}_\beta \ni V \subseteq U_\alpha \in \mathcal{B}_\alpha$, which is absurd. □

Lemma 2.21. *Let p be a point in a space X and let κ be a singular cardinal. If any of the following conditions hold, then* $\operatorname{split}_\kappa(p, X) = \operatorname{split}_{<\kappa}(p, X)$.

(1) $\operatorname{split}_\kappa(p, X)$ *is a limit cardinal.*
(2) $\operatorname{split}_{\operatorname{cf}\kappa}(p, X) \leq \operatorname{cf}(\operatorname{split}_{<\kappa}(p, X))$.
(3) $\operatorname{split}_{<\kappa}(p, X)$ *is regular.*
(4) $\operatorname{cf}(\operatorname{split}_{<\kappa}(p, X)) > \operatorname{cf}\kappa$.
(5) $\kappa = \chi(p, X)$ *and* $\operatorname{cf}(\operatorname{split}_{<\kappa}(p, X)) \geq \operatorname{cf}\kappa$.

Proof. By Theorem 2.18, (1) $\Rightarrow$ $\mathrm{split}_\kappa(p,X)=\mathrm{split}_{<\kappa}(p,X)$. By Lemma 2.15 with $\lambda=\mathrm{split}_{<\kappa}(p,X)$, (2) also implies that $\mathrm{split}_\kappa(p,X)=\mathrm{split}_{<\kappa}(p,X)$. (3) implies that

$$\mathrm{split}_{\mathrm{cf}\,\kappa}(p,X)\leq\mathrm{split}_{<\kappa}(p,X)=\mathrm{cf}(\mathrm{split}_{<\kappa}(p,X));$$

(4) implies that $\mathrm{split}_{\mathrm{cf}\,\kappa}(p,X)\leq(\mathrm{cf}\,\kappa)^+\leq\mathrm{cf}(\mathrm{split}_{<\kappa}(p,X))$. Thus, (3) and (4) each imply (2). Finally, by Lemma 2.20, (5) also implies (2). □

Theorem 2.22. *Let p be a point in a space X. Given any two infinite cardinals $\lambda<\kappa$, $\mathrm{split}^\lambda_\kappa(p,X)=\omega$ if and only if $\mathrm{split}_\kappa(p,X)=\omega$. Hence, if p is flat in X if and only if $\Delta_I(p)$ is flat in X^I for all I.*

Proof. For κ regular, apply Lemma 2.12. For κ singular, apply Theorem 2.16 and case (1) of Lemma 2.21. For the second half of the corollary, first note that we may assume that p is not isolated in X. Second, note that we may assume I is infinite by Theorem 2.10. Finally, apply Corollary 2.7 if $|I|\geq\chi(p,X)$, and otherwise apply the first half of this corollary with $\kappa=\chi(p,X)$ and $\lambda=|I|$. □

The next example shows that $\mathrm{split}_\kappa(p,X)=\mathrm{split}_{<\kappa}(p,X)$ is possible when condition (2) of Lemma 2.21 fails.

Example 2.23. Let $p\in X=\prod^{(\aleph_\omega)}_{\alpha<\aleph_{\omega_1}}2$. By Theorem 2.6, we have $\aleph_\omega=\mathrm{split}_{<\aleph_{\omega_1}}(p,X)$, $\mathrm{split}_{\aleph_1}(p,X)=\aleph_2$, and $\mathrm{split}_{\aleph_{\omega_1}}(p,X)=\aleph_\omega$.

Example 2.17 and the next example show that when condition (2) of Lemma 2.21 fails, $\mathrm{split}_\kappa(p,X)=\mathrm{split}_{<\kappa}(p,X)^+$ is also possible.

Example 2.24. Let $X=\square_{n<\omega}(\omega_{n+1}+1)$ and $p=\langle\omega_{n+1}\rangle_{n<\omega}$. Since p is a P-point in X, $\mathrm{split}_\omega(p,X)=\aleph_1$. For each $n<\omega$, $\mathrm{split}_{\aleph_{n+1}}(p,X)\leq\aleph_{n+1}$ because $\{\{q\in X:q(n)>\alpha\}:\alpha<\omega_{n+1}\}$ is $\aleph_{n+1}$-splitting at p. Let us show that $\mathrm{split}_{\aleph_{n+1}}(p,X)$ actually equals $\aleph_{n+1}$. Let $\langle A_\alpha\rangle_{\alpha<\omega_{n+1}}$ be a sequence of neighborhoods of p. There then exist $I\in[\omega_{n+1}]^{\aleph_n}$ and $s\in\prod_{i<n}\omega_{i+1}$ such that for each $\alpha\in I$, there exists $f_\alpha\in\prod_{i<\omega}\omega_{i+1}$ such that $\prod_{i<\omega}(f_\alpha(i),\omega_{i+1}]\subseteq A_\alpha$ and $s\subseteq f_\alpha$. For each $i<n$, set $g(i)=s(i)$. For each $i\in[n,\omega)$, set $g(i)=\sup_{\alpha\in I}f_\alpha(i)$. We then have $p\in\prod_{i<\omega}(g(i),\omega_{i+1}]\subseteq A_\alpha$ for all $\alpha\in I$, so $\langle A_\alpha\rangle_{\alpha<\omega_{n+1}}$ is not $\aleph_n$-splitting, as desired.

It follows that $\mathrm{split}_{<\aleph_\omega}(p,X)=\aleph_\omega$. Notice that $\mathrm{cf}(\mathrm{split}_{<\aleph_\omega}(p,X))<\mathrm{split}_\omega(p,X)$. Let us show that $\mathrm{split}_{\aleph_\omega}(p,X)=\aleph_{\omega+1}$. Let $\langle B_\alpha\rangle_{\alpha<\aleph_\omega}$ be a sequence of neighborhoods of p in X. For each $n<\omega$, we repeat an argument from the previous paragraph to get an $I_n\in[\aleph_\omega]^{\aleph_n}$ and a $g_n\in\prod_{i<\omega}\omega_{i+1}$ such that $p\in\prod_{i<\omega}(g_n(i),\omega_{i+1}]\subseteq B_\alpha$ for all $\alpha\in I_n$. Setting $J=\bigcup_{n<\omega}I_n$ and $h(i)=\sup_{n<\omega}g_n(i)$ for all $i<\omega$, we have $p\in\prod_{i<\omega}(h(i),\omega_{i+1}]\subseteq B_\alpha$ for all $\alpha\in J$, so $\langle B_\alpha\rangle_{\alpha<\omega_{n+1}}$ is not $\aleph_\omega$-splitting, as desired.

In contrast, it is easy to check that if $X=\square_{n<\omega}(\omega_n+1)$, then we still have $\mathrm{split}_{<\aleph_\omega}(p,X)=\aleph_\omega$, but $\mathrm{split}_\omega(p,X)=\omega$, so $\mathrm{split}_{\aleph_\omega}(p,X)=\aleph_\omega$.

Lemma 2.25. *If $p\in X=\prod_{i\in I}X_i$, then*

$$\chi\mathrm{Nt}(p,X)\leq\sup_{i\in I}\chi\mathrm{Nt}(p(i),X_i).$$

Hence, $\chi\mathrm{Nt}(X) \leq \sup_{i\in I} \chi\mathrm{Nt}(X_i)$.

The $\mathrm{Nt}(X)$-version of the above lemma is true and was first proved by Peregudov [17]. The above version is from [15, Theorem 2.2], but both versions are proved in the same way.

Theorem 2.26. *For all spaces X, $\chi\mathrm{Nt}(X^\omega) = \chi\mathrm{Nt}(X)$. Moreover,*

$$\chi\mathrm{Nt}(p, X) = \chi\mathrm{Nt}^\omega(p, X)$$

for all $p \in X$.

Proof. By Lemma 2.25, $\chi\mathrm{Nt}(X) \geq \chi\mathrm{Nt}(X^\omega)$; let us show that $\chi\mathrm{Nt}(X) \leq \chi\mathrm{Nt}(X^\omega)$. Fix $p \in X$ and set $\kappa = \chi(p, X)$. If $\kappa = \omega$, then

$$\chi\mathrm{Nt}(p, X) = \omega = \chi\mathrm{Nt}^\omega(p, X) \leq \chi\mathrm{Nt}(X^\omega).$$

If $\operatorname{cf}\kappa > \omega$, then

$$\chi\mathrm{Nt}(p, X) = \mathrm{split}_\kappa(p, X) = \mathrm{split}^\omega_\kappa(p, X) = \chi\mathrm{Nt}^\omega(p, X) \leq \chi\mathrm{Nt}(X^\omega)$$

by Lemma 2.12. If $\kappa > \operatorname{cf}\kappa = \omega$, then we have

$$\begin{aligned}\chi\mathrm{Nt}(p, X) = \mathrm{split}_\kappa(p, X) &= \mathrm{split}_{<\kappa}(p, X) \\ &= \mathrm{split}^\omega_\kappa(p, X) = \chi\mathrm{Nt}^\omega(p, X) \leq \chi\mathrm{Nt}(X^\omega)\end{aligned}$$

by case (5) of Lemma 2.21 and Theorem 2.16. Thus, $\chi\mathrm{Nt}(X^\omega) = \chi\mathrm{Nt}(X)$ and $\chi\mathrm{Nt}(p, X) = \chi\mathrm{Nt}^\omega(p, X)$ for all $p \in X$. □

Definition 2.27.

- Let $H(\theta)$ denote the set of all sets hereditarily of size less than θ, where θ is a regular cardinal sufficiently large for the argument at hand.
- Let $M \prec H(\theta)$ mean that $\langle M, \in\rangle$ is an elementary substructure of $\langle H(\theta), \in\rangle$.

To simplify closing-off arguments in this section and in Section 3, we will use elementary substructures. A particularly useful closure property is that if ν is a cardinal, $M \prec H(\theta)$, and $\nu \cap M \in \nu+1$, then $[H(\theta)]^{<\nu} \cap M \subseteq [M]^{<\nu}$.

The next example shows that there are points p in spaces X and singular cardinals κ such that $\kappa = \chi(p, X)$ and $\mathrm{split}_\kappa(p, X) = \mathrm{split}_{<\kappa}(p, X)^+$. In such cases, $\chi\mathrm{Nt}(p, X) = \chi\mathrm{Nt}^{\operatorname{cf}\kappa}(p, X)^+$ by Theorem 2.16. Observe that κ cannot have countable cofinality by Theorem 2.26.

Example 2.28. Let $p \in X = \prod^{(\aleph_1)}_{\alpha<\tau} \prod^{(\aleph_\omega)}_{\beta<\beth_\alpha} 2$ where τ is a regular uncountable cardinal such that τ is not strongly inaccessible and $\operatorname{cf}\left([\tau]^{\aleph_0}\right) = \tau$. For example, τ could be any regular uncountable cardinal of the form $\beth^{+n}_\alpha$ where $n < \omega$ and $\operatorname{cf}\alpha \neq \omega$. For each $\alpha < \tau$, set $X_\alpha = \prod^{(\aleph_\omega)}_{\beta<\beth_\alpha} 2$ and let $\pi_\alpha \colon X \to X_\alpha$ be the natural coordinate projection. Because $\chi(p(\alpha), X_{\alpha+1}) = \operatorname{cf}\left([\beth_{\alpha+1}]^{<\aleph_\omega}\right) = \beth_{\alpha+1}$ for all $\alpha \in [\omega, \tau)$, and $\operatorname{cf}\left([\tau]^{\aleph_0}\right) = \tau$, we have $\chi(p, X) = \beth_\tau$.

Set $\kappa = \beth_\tau$. First, let us show that $\mathrm{split}_{<\kappa}(p, X) = \aleph_\omega$. Fix $\varepsilon < \tau$ such that $\beth_\varepsilon \geq \tau$. Suppose that $\varepsilon \leq \alpha < \tau$ and $\lambda = \beth^+_\alpha$. By Lemma 2.1 and Theorem 2.6, $\mathrm{split}_\lambda(p, X) \leq \mathrm{split}_\lambda(p, X_{\alpha+1}) = \aleph_\omega$. Let us show that

$\mathrm{split}_\lambda(p,X) \geq \aleph_\omega$. Suppose that $n<\omega$ and $\langle A_\alpha\rangle_{\alpha<\lambda}$ is a sequence of neighborhoods of p. We may assume that each A_α is a basic open set, by which we mean a countably supported product of $(<\aleph_\omega)$-supported boxes. Since $\mathrm{cf}\,\lambda > \tau\cdot\beth_1$, there exist $I\in[\lambda]^\lambda$, $s\in[\tau]^{\aleph_0}$, and $f\colon s\to\omega$ such that for each $\alpha\in I$, $\mathrm{supp}\,(A_\alpha)=s$ and, for each $\beta\in s$, $|\mathrm{supp}\,(\pi_\beta[A_\alpha])|\leq\aleph_{f(\beta)}$. Therefore, for all $J\in[I]^{\aleph_n}$, we have $\mathrm{supp}\left(\bigcap_{\alpha\in J}A_\alpha\right)=s$ and, for all $\beta\in s$, $\left|\mathrm{supp}\left(\pi_\beta\left[\bigcap_{\alpha\in J}A_\alpha\right]\right)\right|\leq\aleph_{f(\beta)}\cdot\aleph_n$. Hence, $\bigcap_{\alpha\in J}A_\alpha$ is open. Thus, $\mathrm{split}_\lambda(p,X)=\aleph_\omega$. Hence, $\mathrm{split}_{<\kappa}(p,X)=\aleph_\omega$.

Finally, let us show that $\mathrm{split}_\kappa(p,X)>\aleph_\omega$. Suppose that $\langle B_\alpha\rangle_{\alpha<\kappa}$ is a sequence of neighborhoods of p. As before, we may assume that each B_α is a basic open set. For each $\alpha\in[\varepsilon,\tau)$, choose $I_\alpha\in[[\beth_\alpha,\beth_\alpha^+)]^{\beth_\alpha^+}$, $s_\alpha\in[\tau]^{\aleph_0}$, and $f_\alpha\colon s_\alpha\to\omega$ such that for each $\beta\in I_\alpha$, $\mathrm{supp}\,(B_\beta)=s_\alpha$, and for each $\gamma\in s_\alpha$, $|\mathrm{supp}\,(\pi_\gamma[B_\beta])|\leq\aleph_{f_\alpha(\gamma)}$. For each $\alpha\in[\varepsilon,\tau)$, set $\zeta_\alpha=\sup\{\beta+1:\beta\in s_\alpha\}$. Construct a sequence $\langle\xi_\alpha\rangle_{\alpha<\tau}$ in $[\varepsilon,\tau)$ as follows. Given $\langle\xi_\beta\rangle_{\beta<\alpha}$, set $\eta_\alpha=\sup_{\beta<\alpha}\zeta_{\xi_\beta}$; choose $\xi_\alpha<\tau$ such that $\xi_\alpha>\eta_\alpha$ and $\xi_\alpha\geq\varepsilon$. For each $\alpha<\tau$, we may then choose $J_\alpha\in[I_{\xi_\alpha}]^{\beth_{\xi_\alpha}^+}$ and a basic open W_α such that $\mathrm{supp}\,(\pi_\gamma[B_j])=\mathrm{supp}\,(\pi_\gamma[W_\alpha])$ for all $\gamma<\eta_\alpha$ and $j\in J_\alpha$.

For each $\alpha<\tau$, let $g_\alpha\colon\tau\to\omega$ be an arbitrary extension of f_{ξ_α}; let $t_\alpha\colon\omega\to s_{\xi_\alpha}$ be a surjection. Let $\langle\langle g_\alpha,t_\alpha\rangle\rangle_{\alpha<\tau}\in M\prec H(\theta)$ and let M be countable. Set $\delta=\sup(\tau\cap M)$. Construct an increasing sequence $\langle i_n\rangle_{n<\omega}$ of ordinals in $\tau\cap M$ as follows. Given $\langle i_m\rangle_{m<n}$, set $S_n=\{\alpha<\tau:\forall m,k<n\ g_\alpha(t_{i_m}(k))=g_\delta(t_{i_m}(k))\}$. Since $\delta\in S_n\in M$, it follows by elementarity that $S_n\cap M$ is unbounded in δ. Hence, we may choose $i_n\in S_n\cap M$ such that $i_n>i_m$ for all $m<n$. Thus, for each $\alpha\in\bigcup_{n<\omega}\mathrm{ran}(t_{i_n})$, $g_\delta(\alpha)\geq g_{i_n}(\alpha)$ for cofinitely many $n<\omega$. Hence, there exists $h\colon\bigcup_{n<\omega}\mathrm{ran}(t_{i_n})\to\omega$ that dominates $g_{i_n}\restriction\mathrm{dom}(h)$ for all $n<\omega$.

For each $n<\omega$, choose $K_n\in[J_{i_n}]^{\aleph_n}$. Set $U=\bigcap_{n<\omega}\bigcap_{\alpha\in K_n}B_\alpha$. It suffices to show that U is open. First, observe that U is a product of boxes and that $\mathrm{supp}\,(U)=\mathrm{dom}(h)$, which is countable. Fix $n<\omega$ and $\gamma\in\mathrm{ran}(t_{i_n})$; it suffices to show that $|\mathrm{supp}\,(\pi_\gamma[U])|<\aleph_\omega$. For all $m\in(n,\omega)$ and $\alpha\in K_m$, $\mathrm{supp}\,(\pi_\gamma[B_\alpha])=\mathrm{supp}\,(\pi_\gamma[W_{i_m}])$, which has size at most $\aleph_{h(\gamma)}$. For all $\alpha\in\bigcup_{m\leq n}K_m$, the set $\mathrm{supp}\,(\pi_\gamma[B_\alpha])$ also has size at most $\aleph_{h(\gamma)}$. Hence, $|\mathrm{supp}\,(\pi_\gamma[U])|\leq\aleph_{h(\gamma)}\cdot\aleph_n$. Thus, U is open; hence, $\mathrm{split}_\kappa(p,X)>\aleph_\omega$.

Remark. We could easily replace $\aleph_\omega$ with, say, $\beth_{\varepsilon+\omega}$, in the above example, thereby obtaining the additional inequality $\mathrm{cf}\,\kappa<\chi\mathrm{Nt}(p,X)$.

If κ is not singular, but rather strongly inaccessible, then it is possible, as shown in the next example, that $\mathrm{split}_{<\kappa}(p,X)^+<\mathrm{split}_\kappa(p,X)$.

Example 2.29. There is a point p in a space X such that

$$\mathrm{split}_{<\kappa}(p,X)=\omega<\chi\mathrm{Nt}(p,X)=\mathrm{split}_\kappa(p,X)=\kappa=\chi(p,X).$$

Let $p\in X=\prod_{\alpha<\kappa}^{(\kappa)}2^\alpha$. Since κ is strongly inaccessible, $\chi(p,X)=\kappa$. For all infinite cardinals $\lambda<\kappa$, $\mathrm{split}_\lambda(p,X)\leq\mathrm{split}_\lambda(p(\lambda),2^\lambda)=\omega$ by Lemma 2.1 and Theorem 2.6.

On the other hand, if $\langle A_\alpha\rangle_{\alpha<\kappa}$ is a sequence of open boxes containing p, then either there exists A such that κ-many A_α equal A, in which case

$\langle A_\alpha\rangle_{\alpha<\kappa}$ is not κ-splitting, or, since κ is strongly inaccessible, we may thin out the sequence such that $\langle\zeta_\alpha\rangle_{\alpha<\kappa}$, where $\zeta_\alpha = \sup(\operatorname{supp}(A_\alpha))$, is an increasing sequence. Assuming the latter holds, $\langle A_\alpha\rangle_{\alpha<\kappa}$ is κ-splitting. Let us show that $\langle A_\alpha\rangle_{\alpha<\kappa}$ is not λ-splitting for any $\lambda<\kappa$. So, fix $\lambda<\kappa$. We may assume that each A_α is a product of finitely supported boxes.

By taking the union of an appropriate elementary chain, construct $M \prec H(\theta)$ such that $\langle A_\alpha\rangle_{\alpha<\kappa}\in M$, $\kappa\cap M\in\kappa$, and $\operatorname{cf}(\kappa\cap M)=\lambda^+$. Set $\delta=\kappa\cap M$. For each $\alpha<\delta$, set

$$S(\alpha)=\{\gamma<\kappa : \forall\beta\le\alpha\ \operatorname{supp}(\pi_\beta[A_\gamma])=\operatorname{supp}(\pi_\beta[A_\delta])\}.$$

Since κ is a strong limit cardinal, we have $\mathcal{P}(x)\subseteq M$ for all $x\in[H(\theta)]^{<\kappa}\cap M$. Hence, $S(\alpha)\in M$ for all $\alpha<\delta$. Moreover, $S(\alpha)\nsubseteq M$ because $\delta\in S(\alpha)$; hence, $|S(\alpha)|=\kappa$. By elementarity, $S(\alpha)\cap\delta$ is cofinal in δ, and so is $\langle\zeta_\alpha\rangle_{\alpha<\delta}$.

Let us construct an increasing sequence $\langle\gamma_i\rangle_{i<\lambda^+}$ in δ as follows. Given $i<\lambda^+$ and $\langle\gamma_j\rangle_{j<i}$, set $\alpha_i=\sup_{j<i}\zeta_{\gamma_j}$, which is less than δ, and choose $\gamma_i\in S(\alpha_i)\cap M$ such that $\gamma_i>\gamma_j$ for all $j<i$. Next, set $U=\bigcap_{i<\lambda^+}A_{\gamma_i}$. It suffices to show that U is open. Set $\eta=\sup(\operatorname{supp}(U))$ and observe that $\eta\notin\operatorname{supp}(U)$. Since $\eta\le\delta<\kappa$, it suffices to show that, for all $\beta<\eta$, $\operatorname{supp}(\pi_\beta[U])$ is finite. Fix $\beta<\eta$ and choose the least $i<\lambda^+$ satisfying $\beta\le\alpha_{i+1}$. We then have $\operatorname{supp}(\pi_\beta[U])=\operatorname{supp}(\pi_\beta[A_{\gamma_i}])\cup\operatorname{supp}(\pi_\beta[A_\delta])$, which is finite.

Question 2.30. Can Example 2.29 be modified so as to obtain

$$\operatorname{split}_{<\kappa}(p,X)=\omega<\chi\mathrm{Nt}(p,X)=\operatorname{split}_\kappa(p,X)=\kappa=\chi(p,X).$$

with κ merely weakly inaccesible?

Theorem 2.31. *If $p\in Y$, Y is a dense subspace of a T_3 space X, and κ is an infinite cardinal, then* $\operatorname{split}_\kappa(p,X)=\operatorname{split}_\kappa(p,Y)$.

Proof. Let λ be an infinite cardinal not exceeding κ, let $I\in[\kappa]^\lambda$, let $\langle A_\alpha\rangle_{\alpha<\kappa}$ be a sequence of regular open X-neighborhoods of p, and let $\langle B_\alpha\rangle_{\alpha<\kappa}$ be a sequence of open Y-neighborhoods of p. If $p\in U\subseteq\bigcap_{\alpha\in I}\operatorname{int}_X\operatorname{cl}_X B_\alpha$ and U is open in X, then $p\in U\cap Y\subseteq\bigcap_{\alpha\in I}B_\alpha$. Therefore, $\operatorname{split}_\kappa(p,X)\le\operatorname{split}_\kappa(p,Y)$. If $p\in V\subseteq\bigcap_{\alpha\in I}(A_\alpha\cap Y)$ and V is open in Y, then $p\in\operatorname{int}_X\operatorname{cl}_X V\subseteq\bigcap_{\alpha\in I}A_\alpha$. Therefore, $\operatorname{split}_\kappa(p,Y)\le\operatorname{split}_\kappa(p,X)$. □

Corollary 2.32. *If $p\in Y$ and Y is a dense subspace of a T_3 space X, then* $\chi\mathrm{Nt}(p,X)=\chi\mathrm{Nt}(p,Y)$.

Proof. Observe that $\chi(p,X)=\chi(p,Y)$ and apply Theorem 2.31. □

Remark. By the Theorem 2.31 and its above corollary, since all of our example spaces in this section are $T_{3.5}$, they can be compactified without changing any of the relevant splitting numbers, characters, and local Noetherian types.

3. Applications to power homogeneous compacta

Definition 3.1. Let U be an open neighborhood of a set K in a product space. We say that U is a *simple* neighborhood of K if, for every open V satisfying $K\subseteq V\subseteq U$, we have $\operatorname{supp}(U)\subseteq\operatorname{supp}(V)$.

Lemma 3.2. *If K is a compact subset of a compact product space $X = \prod_{i\in I} X_i$ and U is an open neighborhood of K, then K has a finitely supported simple neighborhood that is contained in U.*

Proof. Set $\sigma = \operatorname{supp}(U)$. By the compactness of K, we may shrink U such that σ is finite. Hence, we may further shrink U until it is minimal in the sense that if V is open and $K \subseteq V \subseteq U$, then $\operatorname{supp}(V)$ is not a proper subset of σ. Suppose that V is open and $K \subseteq V \subseteq U$; set $\tau = \operatorname{supp}(V)$. It then suffices to show that $\sigma \subseteq \tau$. Suppose that $p \in K$, $q \in X$, and $\pi^I_{\sigma\cap\tau}(p) = \pi^I_{\sigma\cap\tau}(q)$. Set $r = (p \restriction \tau) \cup q \restriction (I \setminus \tau)$. We then have $\pi^I_\tau(r) = \pi^I_\tau(p)$, so $r \in V \subseteq U$. Moreover, $\pi^I_\sigma(q) = \pi^I_\sigma(r)$, so $q \in U$. Thus, $(\pi^I_{\sigma\cap\tau})^{-1}\left[\pi^I_{\sigma\cap\tau}[K]\right] \subseteq U$. By the Tube Lemma, there is an open W such that $K \subseteq W \subseteq U$ and $\operatorname{supp}(W) \subseteq \sigma \cap \tau$. By minimality of U, the set $\sigma\cap\tau$ is not a proper subset of σ; hence, $\sigma \subseteq \tau$. $\square$

Definition 3.3.

- Let $\operatorname{Aut}(X)$ denote the group of autohomeomorphisms of X.
- Let $C(X)$ denote the algebra of real-valued continuous functions on X.

Lemma 3.4. *Suppose κ is a regular uncountable cardinal and I is a set and $X = \prod_{i\in I} X_i$ is a compactum and $p \in X$ and $h \in \operatorname{Aut}(X)$ and $\operatorname{split}_\kappa(p(i), X_i) \geq \aleph_1$ for all $i \in I$. Further suppose $\{C(X), p, h\} \subseteq M \prec H(\theta)$ and $\kappa \cap M \in \kappa + 1$. We then have*

$$\operatorname{supp}\left(h\left[(\pi^I_{I\cap M})^{-1}\left[\left\{\pi^I_{I\cap M}(p)\right\}\right]\right]\right) \subseteq M.$$

Proof. For each $i \in I$, let $\mathcal{U}_i$ denote the set of open neighborhoods of $p(i)$. For each $U \in \mathcal{U}_i$, let $V(U,i)$ be a finitely supported simple neighborhood of $h\left[\pi_i^{-1}[\{p(i)\}]\right]$ that is contained in $h\left[\pi_i^{-1}[U]\right]$ (using Lemma 3.2); set $\sigma(U,i) = \operatorname{supp}(V(U,i))$. By elementarity, we may assume that the map V is in M, so $\sigma \in M$ too. Let $W(U,i)$ be an open neighborhood of $p(i)$ such that $\pi_i^{-1}[W(U,i)] \subseteq h^{-1}[V(U,i)]$.

Fix $j \in I$. Suppose $\left|\bigcup_{U\in\mathcal{U}_j} \sigma(U,j)\right| \geq \kappa$. There then exists $\langle U_\alpha\rangle_{\alpha<\kappa} \in \mathcal{U}_j^\kappa$ such that $\sigma(U_\alpha, j) \not\subseteq \sigma(U_\beta, j)$ for all $\beta < \alpha < \kappa$. Fix $E \in [\kappa]^\omega$ and an open neighborhood H of $h\left[\pi_j^{-1}[\{p(j)\}]\right]$ with finite support τ. Choose $\alpha \in E$ such that $\sigma(U_\alpha, j) \not\subseteq \tau$. By simplicity, $H \not\subseteq V(U_\alpha, j)$. Thus, $h\left[\pi_j^{-1}[\{p(j)\}]\right] \not\subseteq \operatorname{int}\bigcap_{\alpha\in E} V(U_\alpha, j)$; hence,

$$\pi_j^{-1}[\{p(j)\}] \not\subseteq \operatorname{int}\bigcap_{\alpha\in E} h^{-1}\left[V(U_\alpha, j)\right] \supseteq \operatorname{int}\bigcap_{\alpha\in E} \pi_j^{-1}[W(U_\alpha, j)];$$

hence, $p(j) \notin \operatorname{int}\bigcap_{\alpha\in E} W(U_\alpha, j)$. Since E was arbitrary, $\{W(U_\alpha, j) : \alpha < \kappa\}$ is ω-splitting at $p(j)$, in contradiction with $\operatorname{split}_\kappa(p(j), X_j) \geq \aleph_1$. Thus,

$$\left|\bigcup_{U\in\mathcal{U}_j} \sigma(U,j)\right| < \kappa.$$

Hence, for each $i \in I \cap M$, we have $\bigcup_{U \in \mathcal{U}_i} \sigma(U,i) \in [I]^{<\kappa} \cap M \subseteq \mathcal{P}(M)$; hence,

$$\operatorname{supp}\left(h\left[(\pi^I_{I\cap M})^{-1}\left[\left\{\pi^I_{I\cap M}(p)\right\}\right]\right]\right) \subseteq \bigcup_{i\in I\cap M}\bigcup_{U\in\mathcal{U}_i} \sigma(U,i) \subseteq M$$

as desired. □

The following theorem is a more precise version of Lemma 1.6.

Theorem 3.5 ([15, Theorem 5.2])**.** *Let X be a compactum and κ an infinite cardinal. Suppose $\pi\chi(p,X) \geq \kappa$ for all $p \in X$. We then have $\operatorname{split}_\kappa(p,X) = \omega$ for some $p \in X$.*

Corollary 3.6. *Let X be a compactum and κ an infinite cardinal. Suppose F is a closed subset of X and $\chi(F,X) < \kappa$ and $\pi\chi(p,X) \geq \kappa$ for all $p \in F$. We then have $\operatorname{split}_\kappa(p,X) = \omega$ for some $p \in F$.*

Proof. Since $\pi\chi(p,X) \leq \pi\chi(p,F)\chi(F,X)$ for all $p \in F$, we have $\pi\chi(p,F) \geq \kappa$ for all $p \in F$. Apply Theorem 3.5 to F. □

The following theorem is an easy generalization of Ridderbos' Lemma 2.2 in [20].

Theorem 3.7. *Suppose X is a power homogeneous Hausdorff space, κ is a regular uncountable cardinal, and D is a dense subset of X such that $\pi\chi(d,X) < \kappa$ for all $d \in D$. We then have $\pi\chi(p,X) < \kappa$ for all $p \in X$.*

Theorem 3.8. *Let κ be a regular uncountable cardinal, X be a power homogeneous compactum, and D be a dense subset of X of size less than κ. Suppose $\operatorname{split}_\kappa(d,X) \geq \aleph_1$ for all $d \in D$. We then have $\operatorname{split}_\kappa(p,X) = \operatorname{split}_\kappa(q,X)$ for all $p,q \in X$. Moreover, $\pi w(X) < \kappa$.*

Proof. Let us first show that $\operatorname{split}_\kappa(p,X) = \operatorname{split}_\kappa(q,X)$ for all $p,q \in X$. Fix $p,q \in X$ such that $\operatorname{split}_\kappa(p,X) \geq \aleph_1$ and $\operatorname{split}_\kappa(q,X) = \min_{x\in X} \operatorname{split}_\kappa(x,X)$. It then suffices to show that that $\operatorname{split}_\kappa(p,X) = \operatorname{split}_\kappa(q,X)$. By Lemmas 2.1 and 2.12, it suffices to show that there exist $A \in [I]^{<\kappa}$ and $f\colon X^A \to X^A$ such that $f(\Delta_A(p)) = \Delta_A(q)$ and f is continuous at $\Delta_A(p)$ and open at $\Delta_A(p)$. Choose I and $h \in \operatorname{Aut}(X^I)$ such that $h(\Delta_I(p)) = h(\Delta_I(q))$. Fix $M \prec H(\theta)$ such that $|M| < \kappa$, $\kappa \cap M \in \kappa$, and $\{C(X), D, h, p\} \subseteq M$. Set $A = I \cap M$ and $Y = X^A \times \{p\}^{I\setminus A} \cong X^A$. Set $f = \pi^I_A \circ (h \restriction Y)$, which is continuous. Since $f(\Delta_I(p)) = \Delta_A(q)$, it suffices to show that f is open at $\Delta_I(p)$.

Fix a closed neighborhood $C \times \{p\}^{I\setminus A}$ of $\Delta_I(p)$ in Y. By the Tube Lemma and Lemma 3.4, there is an open neighborhood U of $\Delta_A(q)$ in X^A such that $(\pi^I_A)^{-1}U \subseteq h\left[(\pi^I_A)^{-1}[C]\right]$. Hence, it suffices to show that $U \subseteq f\left[C \times \{p\}^{I\setminus A}\right]$. Set

$$E = \bigcup\left\{D^\sigma \times \{p\}^{I\setminus\sigma} : \sigma \in [I]^{<\omega}\right\}$$

and $Z = \pi^I_A[E] \times \{p\}^{I\setminus A} = E \cap M$. We then have $\pi^I_A[Z]$ is dense in X^A. Fix $z \in \pi^I_A[Z] \cap U$. By Lemma 3.4 applied to h^{-1} and $z \cup \Delta_{I\setminus A}(p)$, we have $\operatorname{supp}\left(h^{-1}\left[(\pi^I_A)^{-1}[\{z\}]\right]\right) \subseteq A$; hence, for all $x \in \pi^I_A\left[h^{-1}\left[(\pi^I_A)^{-1}[\{z\}]\right]\right] \subseteq$

C, we have $f(x \cup \Delta_{I\setminus A}(p)) = z$. Thus, $\pi_A^I[Z] \cap U \subseteq f\left[C \times \{p\}^{I\setminus A}\right]$. Hence, $U \subseteq \overline{f\left[C \times \{p\}^{I\setminus A}\right]} = f\left[C \times \{p\}^{I\setminus A}\right]$.

Thus, $\mathrm{split}_\kappa(p, X) = \mathrm{split}_\kappa(q, X) \geq \aleph_1$ for all $p, q \in X$. By Corollary 3.6, X has no closed G_δ subset K for which $\pi\chi(p, X) \geq \kappa$ for all $p \in K$. Hence, X has no open subset U for which $\pi\chi(p, X) \geq \kappa$ for all $p \in U$. By Theorem 3.7, $\pi\chi(p, X) < \kappa$ for all $p \in X$. Hence, $\pi w(X) \leq \sum_{d\in D} \pi\chi(d, X) < \kappa$. □

Corollary 3.9. *Let D be a dense subset of a power homogeneous compactum X and let κ be a regular uncountable cardinal. Suppose* $\max_{p\in X} \chi(p, X) = \kappa$*, $|D| < \kappa$, and $\chi\mathrm{Nt}(d, X) \geq \aleph_1$ for all $d \in D$. We then have $\pi w(X) < \chi(p, X) = \kappa$ and $\chi\mathrm{Nt}(p, X) = \chi\mathrm{Nt}(X)$ for all $p \in X$.*

Proof. Each $d \in D$ either has character κ, in which case $\mathrm{split}_\kappa(d, X) = \chi\mathrm{Nt}(d, X) \geq \aleph_1$, or it has character less than κ, in which case

$$\mathrm{split}_\kappa(d, X) = \kappa^+ \geq \aleph_1.$$

By Theorem 3.8, $\mathrm{split}_\kappa(p, X) = \mathrm{split}_\kappa(q, X)$ for all $p, q \in X$ and $\pi w(X) < \kappa$. If $\mathrm{split}_\kappa(X) = \kappa^+$, then no point of X has character κ, which is absurd. Hence, $\mathrm{split}_\kappa(X) \leq \kappa$; hence, every point of X has character at least κ; hence, every point has character κ; hence, $\chi\mathrm{Nt}(p, X) = \mathrm{split}_\kappa(X)$ for all $p \in X$. □

Corollary 3.10 (GCH)**.** *There do not exist X, D, and κ as in the previous corollary. Hence, if X is a power homogeneous compactum and* $\max_{p\in X} \chi(p, X) = \mathrm{cf}\, \chi(X) > d(X)$*, then there is a nonempty open $U \subseteq X$ such that $\chi\mathrm{Nt}(p, X) = \omega$ for all $p \in U$.*

Proof. Seeking a contradiction, suppose X, D, and κ are as in the previous corollary. By Proposition 2.1 of [20], $2^{\chi(Y)} \leq 2^{\pi\chi(Y)c(Y)}$ for every power homogeneous compactum Y. Hence, by GCH, $\kappa \leq \pi\chi(X)c(X)$. Since $\pi\chi(X) \leq \pi w(X) < \kappa$, it follows that $\kappa \leq c(X)$. Hence, $\kappa \leq c(X) \leq \pi w(X) < \kappa$, which is absurd. □

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