Reflecting cones on boolean algebras

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• A poset $P$ is $\kappa^{op}$-like if $\forall x \in P \ |\uparrow x| = |\{y \in P : y \geq x\}| < \kappa$.

• A base of a space $X$ is a family $\mathcal{B}$ of open sets such that
  \[ \forall p \in X \ \forall U \ \text{open} \ \exists p \ \exists V \in \mathcal{B} \ p \in V \subseteq U. \]

• For our purposes, all bases are ordered by inclusion. Also, all spaces are Hausdorff.

• The weight $w(X)$ of $X$ is
  \[ \min\{\kappa \geq \omega : \exists \mathcal{B} \ \text{base of} \ X \ |\mathcal{B}| \leq \kappa\}. \]

  The order weight $ow(X)$ of $X$ is
  \[ \min\{\kappa \geq \omega : \exists \mathcal{B} \ \text{base of} \ X \ \mathcal{B} \text{ is } \kappa^{op}\text{-like}\}. \]
Example. Suppose $X$ is a compact metric space. For each $n < \omega$, let $B_n$ be a finite cover by balls of radius $2^{-n}$. Then $\bigcup_{n<\omega} B_n$ is an $\omega^{\text{op}}$-like base of $X$; hence, $ow(X) = \omega$. 
• The **cellularity** $c(X)$ of $X$ is

$$\sup\left(\{\omega\} \cup \{\kappa : X \text{ has } \kappa\text{-many disjoint open sets}\}\right).$$

• **van Douwen’s Problem.** $c(X) \leq 2^{|\mathbb{N}_0|}$ for all known homogeneous compact $X$. Is there a counterexample? (After well over twenty years, this is still open in all models of ZFC.)

• Similarly, $ow(X) \leq (2^{|\mathbb{N}_0|})^+$ for all known homogeneous compact $X$. Is there a counterexample? (After almost one year, this is still open in all models of ZFC.)

• Is there a connection between $c(X)$ and $ow(X)$?
• A compact space $X$ is *dyadic* if it is a continuous image of $2^\kappa$ for some $\kappa$.

• $c(X) = \omega$ for all compact dyadic $X$.

• $ow(X)$ can be arbitrarily large for compact dyadic $X$, but $ow(X) \neq \omega_1$. If $X$ is also homogeneous, then $ow(X) = \omega$.

• In particular, $ow(G) = \omega$ for every compact group $G$, for every compact group is homogeneous and dyadic (Kuzminov, 1959).
• A local $\pi$-base at a point $p$ in a space $X$ is a family $\mathcal{B}$ of open sets such that

$$\forall U \text{ open } \exists \ p \ \exists V \in \mathcal{B} \ 0 \neq V \subseteq U.$$ 

The $\pi$-character $\pi\chi(p, X)$ of $p$ is

$$\min\{\kappa \geq \omega : \exists \mathcal{B} \text{ local } \pi\text{-base at } p \ |\mathcal{B}| \leq \kappa\}.$$ 

• If $X$ is homogeneous, compact, and dyadic, then $\pi\chi(p, X) = \omega(X)$ for all $p \in X$ (Gerlits, 1976).

• **Theorem 1.** $\omega(X) \neq \omega_1$ for all compact dyadic $X$. Moreover, if $\pi\chi(p, X) = \omega(X)$ for all $p \in X$, then $\omega(X) = \omega$ and every base of $X$ contains an $\omega^{\text{op}}$-like base.
Does Theorem 1 hold for any class of nondyadic compact spaces?

- A subset $I$ of a boolean algebra is *independent* if, given any two disjoint finite subsets $\sigma$ and $\tau$ of $I$, we have $\bigwedge \sigma \land \neg \bigvee \tau \neq 0$.

- A boolean algebra is *free* if it is generated by an independent subset.

- A boolean algebra is free iff it is isomorphic to the algebra $\text{Clop}(2^\kappa)$ of clopen subsets of $2^\kappa$ for some $\kappa$. In particular, $\text{Clop}(2^\kappa)$ is generated by the independent subset

$$\{\{f \in 2^\kappa : f(\alpha) = 1\} : \alpha < \kappa\}.$$
• A boolean algebra $B$ reflects cones if, for all sufficiently large regular cardinals $\theta$, there is a countable language $\mathcal{L}$ and an $\mathcal{L}$-expansion $\langle H_\theta, \in, \ldots \rangle$ of $\langle H_\theta, \in \rangle$ such that

$$\forall M \prec_{\mathcal{L}} H_\theta \quad \forall p \in B \quad \exists \min(M \cap \uparrow p).$$

• Every free boolean algebra reflects cones.

• Denote the Stone dual of a boolean algebra $B$ (i.e., the space of ultrafilters of $B$) by $\text{st}(B)$.

Example: $\text{st}(\text{Clop}(2^\kappa)) \cong 2^\kappa$. 
• **Theorem 2.** Suppose $B$ reflects cones and $X$ is a continuous image of $\text{st}(B)$. Then $\omega(X) \neq \omega_1$. Moreover, if $\pi_X(p, X) = w(X)$ for all $p \in X$, then $\omega(X) = \omega$ and every base of $X$ contains an $\omega^{\text{op}}$-like base.

• Suppose $A$ and $B$ be boolean algebras. Then $\text{st}(B)$ is a a continuous image of $\text{st}(A)$ iff $B$ is isomorphic to a subalgebra of $A$.

• Therefore, Theorem 2 is strictly stronger than Theorem 1 iff there exists a boolean algebra $B$ such that

\[ (*) \quad B \text{ reflects cones but is not a subalgebra of a free boolean algebra.} \]
Is (*) ever satisfied? Not by boolean algebras of size $\leq \kappa_1$. For larger boolean algebras, we have only partial results.

- A boolean algebra $B$ $n$-reflects cones if, for all sufficiently large regular cardinals $\theta$, there is a countable language $\mathcal{L}$ such that given any $p \in B$ and $\in$-chain $M_0, \ldots, M_{n-1}$ satisfying $M_i \prec_{\mathcal{L}} H_\theta$ for all $i < n$, there exists $\min(A \cap \uparrow p)$, where $A$ is a subalgebra of $B$ generated by $B \cap \bigcup_{i<n} M_i$.

- Free boolean algebras $n$-reflect cones for all $n < \omega$.

- If $B$ $n$-reflects cones and $|B| \leq \kappa_n$, then $B$ is a subalgebra of a free boolean algebra. If $B$ $n$-reflects cones for all $n < \omega$, then $B$ is a subalgebra of a free boolean algebra.
In proving our results, the following lemma, which is based on a technique of Jackson and Mauldin, is heavily used.

**Lemma.** Let $\mathcal{L}$ be a countable language, $\beta$ an ordinal, $\theta$ a sufficiently large regular cardinal, and $\langle H_\theta, \in, \ldots \rangle$ an $\mathcal{L}$-expansion of $\langle H_\theta, \in \rangle$. Let $\langle M_\alpha \rangle_{\alpha < \beta}$ satisfy

$$|M_\alpha| = \aleph_0 \text{ and } \langle M_\delta \rangle_{\delta < \alpha} \in M_\alpha \prec \mathcal{L} H_\theta$$

for all $\alpha < \beta$. Then, for each $\alpha < \beta$, there is a finite $\in$-chain $N_0, \ldots, N_{k-1}$ such that

$$\bigcup_{i < k} N_i = \bigcup_{\delta < \alpha} M_\delta \text{ and } \forall i < k \ M_\alpha \ni N_i \prec \mathcal{L} H_\theta.$$ 

Moreover, if $\beta \leq \omega_{n+1}$, then we can get $k \leq n + 1$. 

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References

