The (λ, κ)-FN and the order theory of bases in boolean algebras

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June 2, 2010
BLAST

The κ-Freese-Nation property

Definition (Fuchino, Koppelberg, Shelah)

- A boolean algebra B has the κ-**FN** iff there is a κ-**FN map** $f\colon B \to [B]^{<\kappa}$, *i.e.*,

$$\forall\{p \leq q\} \subseteq B \;\; \exists r \in f(p) \cap f(q) \;\; p \leq r \leq q.$$

- The **FN** is the ω-FN.
- The **weak FN** or **WFN** is the ω_1-FN.

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Basic facts

- If $|B| \leq \kappa$, then B has the κ-FN.
- The κ-FN is preserved by coproducts.
- In particular, free boolean algebras have the FN.
- The interval algebra of κ^+ lacks the κ-FN.

The κ-FN in terms of elementary submodels

- Let $\kappa = \operatorname{cf}\kappa \geq \omega$.
- Let $\theta = \operatorname{cf}\theta$ be sufficiently large.
- $H_\theta = \{x : |x|, |\bigcup x|, |\bigcup\bigcup x|, \ldots < \theta\}$.

Small-model version (FKS)

B has the κ-FN iff, for every $p \in B$ and every M with

$$|M| = \kappa \subseteq M$$

and $B \in M \prec H_\theta$, we have $\operatorname{cf}(M \cap {\downarrow}p) < \kappa$ and $\operatorname{ci}(M \cap {\uparrow}p) < \kappa$.

Big-model version

B has the κ-FN iff, for every $p \in B$ and every M with

$$\kappa \subseteq M$$

and $B \in M \prec H_\theta$, we have $\operatorname{cf}(M \cap {\downarrow}p) < \kappa$ and $\operatorname{ci}(M \cap {\uparrow}p) < \kappa$.

Between the FN and the κ-FN

Assume $\lambda = \operatorname{cf} \lambda > \kappa = \operatorname{cf} \kappa \geq \omega$.

Small-model definition

A boolean algebra B has the (λ, κ)-FN iff, for every $p \in B$ and every M with

$$\lambda \cap M \in \lambda > |M|$$

and $B \in M \prec H_\theta$, we have $\operatorname{cf}(M \cap {\downarrow} p) < \kappa$ and $\operatorname{ci}(M \cap {\uparrow} p) < \kappa$.

Equivalent big-model definition

A boolean algebra B has the (λ, κ)-FN iff, for every $p \in B$ and every M with

$$\lambda \subseteq M \text{ or } \lambda \cap M \in \lambda > |M|$$

and $B \in M \prec H_\theta$, we have $\operatorname{cf}(M \cap {\downarrow} p) < \kappa$ and $\operatorname{ci}(M \cap {\uparrow} p) < \kappa$.

Properties of the (λ, κ)-FN

- Every boolean algebra with cardinality $< \lambda$ has the (λ, ω)-FN, which implies the (λ, κ)-FN for all regular $\kappa \in [\omega, \lambda)$.
- The (λ, κ)-FN is preserved by coproducts.

Properties of the (λ, κ)-FN

- Every boolean algebra with cardinality $< \lambda$ has the (λ, ω)-FN, which implies the (λ, κ)-FN for all regular $\kappa \in [\omega, \lambda)$.
- The (λ, κ)-FN is preserved by coproducts.
- The κ-FN is equivalent to the (κ^+, κ)-FN.
- In particular, the FN and WFN are equivalent to the (ω_1, ω)-FN and (ω_2, ω_1)-FN.
- The algebra of $(< \kappa)$-supported, constructible subsets of ${}^{\lambda}2$ has the (λ, κ)-FN, but lacks the (λ, μ)-FN for all regular $\mu < \kappa$.
- Hence, the implications κ-FN $\Rightarrow$ (κ^+, ω)-FN $\Rightarrow$ FN are all strict if $\kappa > \omega$. (*E.g.*, WFN $\not\Leftarrow$ (ω_2, ω)-FN $\not\Leftarrow$ FN.)

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- The algebra of $(< \kappa)$-supported, constructible subsets of ${}^{\lambda}2$ has the (λ, κ)-FN, but lacks the (λ, μ)-FN for all regular $\mu < \kappa$.
- Hence, the implications κ-FN $\Rightarrow (\kappa^+, \omega)$-FN $\Rightarrow$ FN are all strict if $\kappa > \omega$. (*E.g.*, WFN $\not\Leftarrow (\omega_2, \omega)$-FN $\not\Leftarrow$ FN.)
- If $\lambda \leq 2^\kappa$, then $\mathcal{P}(\kappa)$ and $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ lack the (λ, κ)-FN, so $2^\omega = \omega_2$ does not decide whether $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega)/[\omega]^{<\omega}$ have the WFN, but it does imply they lack the (ω_2, ω)-FN.

Elementary quotients

Definition

Given a compactum X and $X \in M \prec H_\theta$, define the quotient map $\pi_M^X \colon X \to X/M$ by $p/M = q/M$ iff $f(p) = f(q)$ for all $f \in C(X, \mathbb{R}) \cap M$.

Basic facts and examples

- The sets of the form U/M where U is open F_σ and $U \in M$ form a base of X/M. (Also, $\bigcup(U/M) = U$ for such U.)
- If X is a 0-dimensional compactum, then $X/M \cong Ult(Clop(X) \cap M)$.
- If X is a compactum, $X \in M \prec H_\theta$, and M is countable, then X/M is metrizable.
- If X is a compact metric space and $X \in M \prec H_\theta$, then $X/M \cong X$.
- If $X = \omega_1 + 1$ and $M \prec H_\theta$ is countable, then $X/M \cong \delta + 1$ where $\delta = \omega_1 \cap M$.

The Stone dual of the FN

- A map between spaces is **open** if open sets have open images.
- *Small-model definition* (Ščepin, Bandlow). A compactum X is said to be **openly generated** iff π^X_M is an open map for a club of M in $[H_\theta]^\omega$.
- *Big-model definition.* A compactum X is openly generated iff π^X_M is an open map for all $M \prec H_\theta$ with $X \in M$.
- A 0-dimensional compactum X is openly generated iff $Clop(X)$ has the FN.

- Example: If $X = \omega_1 + 1$, $M \prec H_\theta$ is countable, and S is the set of countable successor ordinals, then S is open but S/M is not open, so π^X_M is not open.

Which compacta are openly generated?

Compare being openly generated to being a continuous image of a product of metrizable compacta (*i.e.*, dyadic):

Theorem (Ščepin)

The class of openly generated compacta includes all metrizable compacta and is closed with respect to products and hyperspaces (with the Vietoris topology).

Theorem (Šapiro)

The hyperspace $exp({}^{\omega_2}2)$ is not a continuous image of a product of metrizable compacta.

Theorem (Engelking, Ščepin)

The quotient of ${}^{\omega_1}2$ formed by identifying $\langle 0\rangle_{i<\omega_1}$ and $\langle 1\rangle_{i<\omega_1}$ is not openly generated.

Stone dual of the (λ, κ)-FN

- A map $f\colon X \to Y$ is said to be κ-**open** if, for all open $O \subseteq X$, $f[O]$ is the intersection of $(< \kappa)$-many open sets.
- Hence, the ω-open maps are exactly the open maps.

- *Small-model definition.* A compactum X is (λ, κ)-**openly generated** iff π^X_M is κ-open for a club of M in $[H_\theta]^{<\lambda}$.
- *Big-model definition.* A compactum X is (λ, κ)-openly generated iff π^X_M is κ-open for all $M \prec H_\theta$ with $X \in M$ and either $\lambda \subseteq M$ or $M \cap \lambda \in \lambda > |M|$.
- A 0-dimensional compactum X is (λ, κ)-openly generated iff $Clop(X)$ has the (λ, κ)-FN.

Which compacta are (λ, κ)-openly generated?

Compare being (λ, κ)-openly generated to being a continuous image of a product of small factors:

Theorem

The class of (λ, κ)-openly generated compacta includes all compacta with weight $< \lambda$ and is closed with respect to products and hyperspaces.

Theorem (Šapiro)

The hyperspace *exp* $\left({}^{\lambda^+}2\right)$ is not a continuous image of a product whose factors are all compacta with weight $< \lambda$.

Theorem

The quotient of ${}^{\lambda}2$ formed by identifying $\langle 0 \rangle_{i<\lambda}$ and $\langle 1 \rangle_{i<\lambda}$ is not (λ, κ)-openly generated.

Bases

- A family of open sets is called a **base** iff every open set is a union of sets from the family.
- Hence, a family of clopen sets in a 0-dimensional compactum is a base iff every clopen set is a finite union of sets from the family.
- A subset of a boolean algebra is called a **base** iff every element of the algebra is a finite join of elements of the subset.

- The **weight** $w(X)$ of a infinite T_0 space X is $\min\{|\mathcal{E}| : \mathcal{E}$ is a base$\}$.
- The **weight** of an infinite boolean algebra B is just $|B|$.

Local π-bases

- A family of nonempty open subsets of a space is called a **local π-base** at a point iff every neighborhood of that point contains an element of the family.
- Hence, a family of nonempty clopen subsets of a 0-dimensional compactum is a local π-base iff every clopen neighborhood of the point contains an element of the family..
- A subset of a boolean algebra is called a **local π-base** at an ultrafilter iff every element of the subset is $\neq 0$, and everything in the ultrafilter is $\geq$ something in the subset.
- The **π-character** $\pi\chi(p, X)$ of a point p is

$$\min\{|\mathcal{E}| : \mathcal{E} \text{ is a local } \pi\text{-base at } p\}.$$

- The **π-character** $\pi\chi(U, B)$ of an ultrafilter is

$$\min\{|S| : S \text{ is a local } \pi\text{-base at } U\}.$$

Continuous images and subalgebras

Theorem (Ščepin)

Assuming that:

- Y is an infinite compactum and a continuous image of an openly generated compactum X, and
- B is an infinite subalgebra of a boolean algebra A where A has the FN,

it follows that:

- $w(Y) = \sup_{p \in Y} \pi\chi(p, Y)$ and $|B| = \sup_{U \in Ult(B)} \pi\chi(U, B)$,
- and every regular uncountable cardinal is a caliber of Y and a precaliber of B.

Definition

A regular cardinal ν is a **caliber (precaliber)** of a space (boolean algebra) if every ν-sized open family (subset) has a ν-sized subset that contains a common point (extends to a proper filter).

Continuous images and subalgebras, part II

Theorem

Assuming that:

- Y is a compactum and a continuous image of a (λ, ω)-openly generated compactum X,
- B is a subalgebra of a boolean algebra A where A has the (λ, ω)-FN, and
- $w(Y) \geq \lambda$ and $|B| \geq \lambda$,

it follows that:

- $w(Y) = \sup_{p \in Y} \pi\chi(p, Y)$ and $|B| = \sup_{U \in Ult(B)} \pi\chi(U, B)$,
- and every regular $\mu \geq \lambda$ is a caliber of Y and a precaliber of B.

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Question. If $\kappa = \kappa^{<\kappa}$, can we replace λ and ω in the theorem with κ^+ and κ?

Remark. The real interval algebra has size 2^ω and the (ω_2,ω_1)-FN, yet all its ultrafilters have π-character ω.

Flat vs. top-heavy

Definition

- A **cone** in a poset P is a set of the form $\uparrow p = \{q \in P : q \geq p\}$ where $p \in P$.
- A poset is κ-**top-heavy** if some cone has size $\geq \kappa$.
- A poset is κ^{op}-**like** if every cone has size $< \kappa$.

Example. $\{1/(n+1) : n \in \omega\}$ is ω^{op}-like.
Example. ω is ω_1^{op}-like and ω-top-heavy.

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Example. $\{1/(n+1) : n \in \omega\}$ is ω^{op}-like.
Example. ω is ω_1^{op}-like and ω-top-heavy.
Convention. A family of subsets of a topological space is ordered by $\subseteq$.

Theorem (joint with Spadaro)

If κ is regular and uncountable, X is a compactum with weight $\geq \kappa$, and X has a dense set of points p with $\pi\chi(p, X) < \kappa$, then X does not have a κ^{op}-like base.

Flatness from the $(\mu^+, \mathrm{cf}\,\mu)$-FN

Theorem

Assuming that:

- A has the $(\mu^+, \mathrm{cf}\,\mu)$-FN,
- S is a subset of A,
- B is a subalgebra of A, and
- E is a base of B,

it follows that:

- S has a dense subset that is μ^{op}-like, and,
- if also $\pi\chi(U, B) = |B|$ for all $U \in \mathit{Ult}(B)$, then E includes a μ^{op}-like base F of B.

Remark

If μ is regular, then the $(\mu^+, \mathrm{cf}\,\mu)$-FN is equivalent to the μ-FN.

Flatness from $(\mu^+, \mathrm{cf}\,\mu)$-open generation

- **Theorem.** Assuming that:
  - X is a $(\mu^+, \mathrm{cf}\,\mu)$-openly generated compactum,
  - Y is a compactum and a continuous image of X,
  - $\mathcal{E}$ is a base of Y, and
  - $\mathcal{S} \subseteq \mathcal{P}(Y)$ is such that the interior of every $U \in \mathcal{S}$ includes the closure of some $V \in \mathcal{S}$,

  it follows that:
  - $\mathcal{S}$ has a dense subset that is μ^{op}-like, and,
  - if also $\pi\chi(p, Y) = w(Y)$ for all $p \in Y$, then $\mathcal{E}$ includes a μ^{op}-like base $\mathcal{F}$ of Y.
- **Corollary.** A base of a compact group always includes an ω^{op}-like base of the group.
- **Question.** Are all homogeneous compacta $((2^\omega)^+, \omega)$-openly generated?

Comments on proving the last two theorems

All those "big-model" definitions were (re?)invented in order to prove the last two theorems.

The construction of the μ^{op}-like dense sets uses continuous elementary chains of submodels and induction on the weight of the space or boolean algebra.

On the other hand, each successor stage of the construction of μ^{op}-like bases uses special properties of countable boolean algebras and metrizable compacta to build a countable piece of the base.

To perform constructions of length $\geq \omega_2$ one countable piece at a time, I used a generalization of Jackson-Mauldin trees of elementary submodels (which they used to build a Steinhaus set without assuming CH).

Long κ-approximation sequences

- Assume $\kappa = \operatorname{cf}\kappa > \omega$.
- Let Ω_κ denote the tree of finite sequences of ordinals $\langle \xi_i \rangle_{i<n}$ which satisfy $\kappa \leq |\xi_i| > |\xi_j|$ for all $\{i < j\} \subseteq n$.

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- The is a unique order isomorphism Υ_κ from the ordinals to Ω_κ ordered lexicographically.
- Thus, $\sqsubseteq_\kappa = \Upsilon_\kappa(\subseteq \restriction \Omega_\kappa)$ is a $\{\kappa\}$-definable tree-ordering of the ordinals such that $\alpha \sqsubset_\kappa \beta \Rightarrow \alpha < \beta$ and all branches are finite.

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- A **long κ-approximation sequence** is a sequence $\langle M_\alpha \rangle_{\alpha<\eta}$ or arbitrary ordinal length satisfying $\kappa, \langle M_\beta \rangle_{\beta<\alpha} \in M_\alpha \prec H_\theta$ and $\kappa \cap M_\alpha \in \kappa > |M_\alpha|$.

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- If $\alpha \leq \eta$ and $\{\beta : \beta \sqsubseteq_\kappa \alpha\} = \{\beta_0 \sqsubset_\kappa \cdots \sqsubset_\kappa \beta_m\}$, then $N_i = \bigcup\{M_\gamma : \beta_i \leq \gamma < \beta_{i+1}\}$ satisfies $|N_i| \subseteq N_i \prec H_\theta$.

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- If $\alpha \leq \eta$ and $\{\beta : \beta \sqsubseteq_\kappa \alpha\} = \{\beta_0 \sqsubset_\kappa \cdots \sqsubset_\kappa \beta_m\}$, then $N_i = \bigcup\{M_\gamma : \beta_i \leq \gamma < \beta_{i+1}\}$ satisfies $|N_i| \subseteq N_i \prec H_\theta$.
- Thus, long ω_1-approximation sequences allow one to construct things by adding one new countable piece per stage, each time collecting all the old pieces into a finite union of possibly uncountable elementary substructures.

Using long approximation sequences

For constructing the μ^{op}-like bases, a single long ω_1-approximation sequence $\vec{M}$ does the job if $\mu = \omega$.

If $\mu > \omega$, then the construction uses a long μ^+-approximation sequence $\vec{M}$ built such that each M_α is the union of an ω_1-approximation sequence $\vec{N}_\alpha$ of length μ with $\langle M_\beta \rangle_{\beta<\alpha} \in N_{\alpha,0}$.

The above can be arranged because for all κ, if τ is a cardinal, then $\{\alpha : \alpha \sqsubseteq_\kappa \tau\} = \{0, \tau\}$, so $\bigcup\{N_{\alpha,\gamma} : 0 \leq \gamma < \mu\} \prec H_\theta$.

More questions, and partial answers

Let κ be any infinite cardinal.

- If A and B are boolean algebras whose coproduct has a κ^{op}-like base, must one of A and B have a κ^{op}-like base?
- If X and Y are compacta and $X \times Y$ has a κ^{op}-like base, must one of X and Y have a κ^{op}-like base?
- **Theorem** (joint with Spadaro). There are non-compact X, Y such that each lacks ω^{op}-like bases, but $X \times Y$ has one.

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- **Theorem** (joint with Spadaro). There are non-compact X, Y such that each lacks ω^{op}-like bases, but $X \times Y$ has one.

- If A is a boolean algebra with bases E, F and E is κ^{op}-like, must F contain a κ^{op}-like base of A?
- If X is a compactum with bases $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{E}$ is κ^{op}-like, must $\mathcal{F}$ contain a κ^{op}-like base of X?
- **Theorem.** ${}^{\omega}\omega$ has an ω^{op}-like base and another base that contains no ω^{op}-like base of ${}^{\omega}\omega$.
- **Theorem.** (GCH) If X is a homogeneous compactum with a κ^{op}-like base, then all its bases include κ^{op}-like bases of X.

A question about $\aleph_\omega$

- Let A be the algebra of countably supported subsets of ${}^{\aleph_\omega}2$.
- Is it consistent, relative to large cardinals, that every dense subset of $A \setminus \{\varnothing\}$ is ω_1-top-heavy?
- A natural place to look is a model of Chang's Conjecture at $\aleph_\omega$. (These exist, assuming (roughly) a huge cardinal.)
- **Theorem** (Kojman, Spadaro)
  - $A \setminus \{\varnothing\}$ has a κ^{op}-like dense subset where $\kappa = \mathrm{cf}([\aleph_\omega]^\omega, \subseteq)$ (which is $< \aleph_{\omega_4}$).
  - If we assume $\square_{\aleph_\omega}$, then $A \setminus \{\varnothing\}$ has an ω_1^{op}-like dense subset.