Tukey classes of ultrafilters on ω

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Quasiorders

- **Definition.** A quasiorder is a set with a transitive reflexive relation (denoted by $\leq$ by default). A quasiorder Q is κ-directed if every subset of size less than κ has an upper bound. We abbreviate "ω-directed" with "directed."

- **Definition** The product $P \times Q$ of two quasiorders P and Q is defined by $\langle p_0, q_0 \rangle \leq \langle p_1, q_1 \rangle$ iff $p_0 \leq p_1$ and $q_0 \leq q_1$.

- **Definition.** A subset C of a quasiorder Q is cofinal if for all $q \in Q$ there exists $c \in C$ such that $q \leq c$. The cofinality of Q (written cf(Q)), is defined as follows.

$$\text{cf}(Q) = \min\{|C| : C \text{ cofinal in } Q\}$$

Tukey equivalence

- **Definition.** A directed set P is Tukey reducible to a directed set Q (written $P \leq_T Q$) if there is map from P to Q such that the image of every unbounded set is unbounded. If $P \leq_T Q \leq_T P$, then we say P and Q are Tukey equivalent and write $P \equiv_T Q$.

- **Theorem** (Tukey, 1940). $P \equiv_T Q$ iff P and Q order-embed as cofinal subsets of a common third directed set.

- $P \leq_T Q \Rightarrow \mathsf{cf}(P) \leq \mathsf{cf}(Q)$

- $\forall \alpha, \beta \in \mathsf{O}n \;\; \alpha \leq_T \beta \Leftrightarrow \mathsf{cf}(\alpha) = \mathsf{cf}(\beta)$

- $P \leq_T P \times Q$

- $P \leq_T R \geq_T Q \Rightarrow P \times Q \leq_T R$.

- $P \times P \equiv_T P$

- $P \leq_T \langle [\mathsf{cf}(P)]^{<\omega}, \subseteq \rangle$

- $\forall A, B$ infinite $\langle [A]^{<\omega}, \subseteq \rangle \leq_T \langle [B]^{<\omega}, \subseteq \rangle \Leftrightarrow |A| \leq |B|$

- Given finitely many ordinals $\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}$, we have
$$\prod_{i<m} \alpha_i \leq_T \prod_{i<n} \beta_i \Leftrightarrow \{\mathrm{cf}(\alpha_i) : i < m\} \subseteq \{\mathrm{cf}(\beta_i) : i < n\}.$$

- Every countable directed set is Tukey equivalent to 1 or ω.

- No two of 1, ω, ω_1, $\omega \times \omega_1$, and $\langle [\omega_1]^{<\omega}, \subseteq \rangle$ are Tukey equivalent.

- (Todorčevič, 1985) PFA implies every ω_1-sized directed set is Tukey equivalent to one of the above five orders. This is false under CH because there are at least 2^{ω_1}-many pairwise Tukey inequivalent directed sets of size $\mathfrak{c}$.

Ultrafilters on ω

- **Definition.** Given $A, B \subseteq \omega$, we write $A \supseteq^* B$ iff A almost contains B, *i.e.*, iff $|B \setminus A| < \omega$.

  Definition. If $\mathcal{A} \subseteq [\omega]^\omega$, then we say $\mathcal{A}$ has the strong finite intersection property (SFIP) iff $|\bigcap \sigma| = \omega$ for all $\sigma \in [\mathcal{A}]^{<\omega}$. We say that $B \in [\omega]^\omega$ is a pseudointersection of $\mathcal{A}$ iff $A \supseteq^* B$ for all $A \in \mathcal{A}$.

  Definition. Denote by ω^* the set of nonprincipal ultrafilters on ω. Any $\mathcal{A} \subseteq [\omega]^\omega$ with the SFIP can be extended to some $\mathcal{U} \in \omega^*$.

- Which quasiorders Q are Tukey equivalent to $\langle \mathcal{U}, \supseteq^* \rangle$ for some $\mathcal{U} \in \omega^*$?

Theorem (Dow & Zhou, 1999). $\exists \mathcal{U} \in \omega^* \ \langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$.

Proof (Milovich). A subset $\mathcal{I}$ of $[\omega]^\omega$ is said to be independent if for all disjoint finite $\sigma, \tau \subseteq \mathcal{I}$ we have $\bigcup \sigma \not\supseteq^* \bigcap \tau$. It is known that there exists an independent $\mathcal{A} \subseteq [\omega]^\omega$ of size $\mathfrak{c}$. Let $\mathcal{B}$ denote the set of all complements of pseudointersections of infinite subsets of $\mathcal{A}$. Since $\mathcal{A}$ is independent, $\mathcal{A} \cup \mathcal{B}$ has the SFIP. Hence, we may extend $\mathcal{A} \cup \mathcal{B}$ to some $\mathcal{U} \in \omega^*$.

We have $\langle \mathcal{U}, \supseteq^* \rangle \leq_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$ simply because $\mathrm{cf}(\langle \mathcal{U}, \supseteq^* \rangle) \leq |\mathcal{U}| = \mathfrak{c}$. Hence, it suffices to show that $\langle [\mathcal{A}]^{<\omega}, \subseteq \rangle \leq_T \langle \mathcal{U}, \supseteq^* \rangle$. Given $\sigma \in [\mathcal{A}]^{<\omega}$, set $f(\sigma) = \bigcap \sigma \in \mathcal{U}$. Suppose Ξ is an unbounded subset of $[\mathcal{A}]^{<\omega}$. Then $\bigcup \Xi$ is infinite. If $\{f(\sigma) : \sigma \in \Xi\}$ is bounded with respect to $\supseteq^*$ by some $X \in [\omega]^\omega$, then X is a pseudointersection of $\bigcup \Xi$; hence, $\omega \setminus X \in \mathcal{B}$; hence, $X \notin \mathcal{U}$. Hence, $\{f(\sigma) : \sigma \in \Xi\}$ is unbounded in $\langle \mathcal{U}, \supseteq^* \rangle$. $\Box$

- **Question.** Is it consistent (with ZFC) that

$$\forall \mathcal{U} \in \omega^* \;\; \langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle?$$

- **Theorem** (Shelah, 1982). It is consistent that

$$\forall \mathcal{U} \in \omega^* \;\; \langle \mathcal{U}, \supseteq^* \rangle \text{ is not } \omega_1\text{-directed.}$$

- **Definition.** Let the pseudointerection number, $\mathfrak{p}$, denote the least κ for which there exists $\mathcal{A} \subseteq [\omega]^\omega$ such that $|\mathcal{A}| = \kappa$ and $\mathcal{A}$ has the SFIP but $\mathcal{A}$ has no pseudointersection.

- It's easy to show that $\mathfrak{p} > \omega$. Suppose $\{A_n : n < \omega\}$ has the SFIP. For each $n < \omega$, set $b_n = \min(\bigcap_{i<n} A_i \setminus \{b_i\})$. Then $\{b_n : n < \omega\}$ is a pseudointersection of $\{A_n : n < \omega\}$.

- CH $\Rightarrow$ MA $\Rightarrow \mathfrak{p} = \mathfrak{c}$.

Theorem (classical). $\mathfrak{p} = \mathfrak{c} \Rightarrow \exists \mathcal{U} \in \omega^* \ \ \langle \mathcal{U}, \supseteq^* \rangle \equiv_T \mathfrak{c}$.

Proof. Let $\langle X_\alpha \rangle_{\alpha<\mathfrak{c}}$ be a bijection from $\mathfrak{c}$ to $[\omega]^\omega$. Recursively construct a strictly $\supseteq^*$-increasing sequence $\langle Y_\alpha \rangle_{\alpha<\mathfrak{c}}$ in $[\omega]^\omega$ as follows. Suppose we have $\alpha < \mathfrak{c}$ and $\langle Y_\beta \rangle_{\beta<\alpha}$ is $\supseteq^*$-increasing. Then $\{Y_\beta\}_{\beta<\alpha}$ has the SFIP. Choose a pseudointersection Z of $\{Y_\beta\}_{\beta<\alpha}$. Then choose $W \in \{Z \cap X_\alpha, Z \setminus X_\alpha\}$ such that $|W| = \omega$. Let Y_α be an infinite and coinfinite subset of W.

Set $\mathcal{U} = \bigcup_{\alpha<\mathfrak{c}} \{X \subseteq \omega : Y_\alpha \subseteq X\}$. Then $\mathcal{U}$ is clearly a nonprincipal filter. Moreover, $\mathcal{U}$ is an ultrafilter because $Y_\alpha \subseteq X_\alpha$ or $Y_\alpha \subseteq \omega \setminus X_\alpha$ for all $\alpha < \mathfrak{c}$. Finally $\mathfrak{c} \equiv_T \langle \mathcal{U}, \supseteq^* \rangle$ because $\langle Y_\alpha \rangle_{\alpha<\mathfrak{c}}$ embeds $\mathfrak{c}$ as a cofinal subset of $\mathcal{U}$ (with respect to $\supseteq^*$). $\square$

- **Theorem.** Suppose $\mathfrak{p} = \mathfrak{c}$. If $\omega \leq \mathrm{cf}(\kappa) = \kappa \leq \mathfrak{c}$, then $\exists \mathcal{U} \in \omega^* \;\; \langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\kappa}, \subseteq \rangle$.

- It is known that $\mathfrak{p} = \mathfrak{c} \Rightarrow \mathfrak{c} = \mathrm{cf}(\mathfrak{c}) \Rightarrow \langle [\mathfrak{c}]^{<\mathfrak{c}}, \subseteq \rangle \equiv_T \mathfrak{c}$. Therefore, this theorem generalizes the previous classical result.

- **Question.** Assuming $\mathfrak{p} = \mathfrak{c}$, does the above theorem enumerate all Tukey classes of elements of ω^*? I don't know the answer in any model of $\mathfrak{p} = \mathfrak{c}$.

- **Theorem.** If κ is an infinite cardinal less than $\mathfrak{p}$ and Q is a κ-directed set that is a union of at most κ-many κ^+-directed sets, then $\forall \mathcal{U} \in \omega^* \;\; \langle \mathcal{U}, \supseteq^* \rangle \not\equiv_T \kappa \times Q$. **3/17/2008: I found a bug in the proof for $\kappa > \omega$; I currently only have a correct proof for $\kappa = \omega$.**

- **Corollary.** $\forall\langle\alpha_i\rangle_{i<n} \in \mathrm{O}n^{<\omega}$ $\forall\mathcal{U} \in \omega^*$ $\langle\mathcal{U}, \supseteq^*\rangle \not\equiv_T \omega \times \prod_{i<n}\alpha_i$. **3/17/2008: See note about previous theorem.**

  Proof. We may assume $\mathrm{cf}(\alpha_i) = \alpha_i$ for all $i < n$. Set $\sigma = \{i < n : \alpha_i \leq \omega\}$. Then $\prod_{i<n}\alpha_i$ is a countable union of ω_1-directed sets because it equals the set

$$\bigcup_{f\in\prod_{i\in\sigma}\alpha_i}\Bigg(\{f\} \times \prod_{i\in n\setminus\sigma}\alpha_i\Bigg).$$

- **Corollary.** Suppose $\mathfrak{p} = \mathfrak{c}$. If $2 \leq n < \omega$ and $\langle\kappa_i\rangle_{i<n}$ is a strictly increasing sequence of infinite regular cardinals, then $\forall\mathcal{U} \in \omega^*$ $\langle\mathcal{U}, \supseteq^*\rangle \not\equiv_T \prod_{i<n}\kappa_i$. **3/17/2008: See note about previous theorem.**

Proof. Suppose $\mathcal{U} \in \omega^*$ and $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \prod_{i<n} \kappa_i$. Then $\mathrm{cf}(\langle \mathcal{U}, \supseteq^* \rangle) = \mathrm{cf}\Big(\prod_{i<n} \kappa_i\Big) = \kappa_{n-1}$. Since $\mathrm{cf}(\langle \mathcal{U}, \supseteq^* \rangle) \leq |\mathcal{U}| = \mathfrak{c}$, we have $\kappa_{n-1} \leq \mathfrak{c}$; hence, $\kappa_0 < \mathfrak{c} = \mathfrak{p}$.$\square$

Theorem. Given any two regular uncountable cardinals κ and λ, it is consistent with ZFC that $\beta\omega \setminus \omega$ has a local base Tukey equivalent to $\kappa \times \lambda$.

Proof outline. We may assume $\kappa < \lambda = \mathfrak{c}$. We build a forcing extension with a cofinal subset C of $\kappa \times \lambda$ and an embedding $\langle Y_{\alpha,\beta} \rangle_{\langle \alpha,\beta \rangle \in C}$ into $\langle [\omega]^\omega, \supseteq^* \rangle$ such that $\mathcal{U}$ is an ultrafilter where $\mathcal{U} = \bigcup_{\langle \alpha,\beta \rangle \in C} \{X \subseteq \omega : Y_{\alpha,\beta} \subseteq X\}$. This will yield $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T C \equiv_T \kappa \times \lambda$.

We proceed via a finite support iteration of length $\lambda \cdot \kappa$. At stage $\lambda \cdot \alpha + \beta$ where $\alpha < \kappa$ and $\beta < \lambda$, we have already constructed the restriction of our embedding to $\{\langle \gamma, \delta \rangle \in \kappa \times \lambda : \lambda \cdot \gamma + \delta < \lambda \cdot \alpha + \beta\}$ such that its range has the SFIP and a few other technical properties. We also are have already chosen some $X_{\alpha,\beta} \in [\omega]^\omega$.

We then argue, heavily relying on $\kappa < \mathrm{cf}(\lambda) = \lambda$, that there are arbitrarily large $\rho < \lambda$ for which there is a forcing extension in which our embedding extends to one with $\langle \alpha, \rho \rangle$ in its domain and a subset of either $X_{\alpha,\beta}$ or $\omega \setminus X_{\alpha,\beta}$ in its range, such that the new range still has the SFIP and our other technical properties. (This forcing extension is not at all exotic. We just use the Mathias forcing for the image of $\alpha \times \rho$ by our given embedding.)

Using standard bookkeeping tricks, we ensure that every element of $[\omega]^\omega$ in the final model appears as some $X_{\alpha,\beta}$, thereby guaranteeing that $\mathcal{U} \in \omega^*$. $\square$

References

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