A locally finite characterization of AE(0) and related classes of compacta

David Milovich
Texas A&M International University
david.milovich@tamiu.edu
http://www.tamiu.edu/~dmilovich/

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Stone duality notation

- A *compactum* is a compact Hausdorff space.
- A *boolean space* is a compactum with a clopen base.
- Clop is the contravarient functor from boolean spaces and continuous maps to boolean algebras and homomorphisms.
  - $\mathrm{Clop}(X)$ is $(\{K \subseteq X : K \text{ clopen}\}, \cap, \cup, K \mapsto X \setminus K\})$.
  - $\mathrm{Clop}(f)(K) = f^{-1}[K]$.
- Modulo isomorphism, the inverse of Clop is the functor Ult:
  - $\mathrm{Ult}(A)$ is $\{U \subseteq A : U \text{ ultrafilter}\}$ with clopen base $\{\{U \in \mathrm{Ult}(A) : a \in U\} : a \in A\}$;
  - $\mathrm{Ult}(\phi)(U) = \phi^{-1}[U]$.

Open is dual to relatively complete.

- A boolean subalgebra A of B is called *relatively complete* if every $b \in B$ has a least upper bound in A.
  - Let $A \leq_{\mathsf{rc}} B$ abbreviate "A is relatively complete in B."
- A boolean homomorphism $\phi\colon A \to B$ is called relatively complete if $\phi[A] \leq_{\mathsf{rc}} B$.
- A boolean homomorphism ϕ is relatively complete iff $\mathsf{Ult}(\phi)$ is open.

AE(0) spaces

Definition

A boolean space X is an *absolute extensor of dimension zero*, or *AE(0)* for short, if, for every continuous $f\colon Y \to X$ with $Y \subseteq Z$ boolean, f extends to a continuous $g\colon Z \to X$.

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Given a boolean space X of weight $\leq \kappa$, the following are known to be equivalent:

- X is AE(0).
- X is Dugundji, *i.e.*, a retract of 2^κ.
- $X \times 2^\kappa \cong 2^\kappa$.
- There exists Y such that $X \cong Y \subseteq 2^\kappa$ and, for all $\alpha < \beta < \kappa$, the projection $\pi_{\alpha,\beta}\colon Y \restriction \beta \to Y \restriction \alpha$ is open.
- $\mathrm{Clop}(X)$ has an additive rc-skeleton, *i.e.*, if $n < \omega$, θ is a regular cardinal, and $\mathrm{Clop}(X) \in N_i \prec H(\theta)$ for all $i < n$, then $\left\langle \mathrm{Clop}(X) \cap \bigcup_{i<n} N_i \right\rangle \leq_{\mathrm{rc}} \mathrm{Clop}(X)$.

Multicommutativity

- A *poset diagram* of boolean spaces is pair of sequences $(\vec{X}, \vec{f})$ with
  - $\mathrm{dom}(\vec{X})$ a poset,
  - X_i a boolean space for all $i \in \mathrm{dom}(\vec{X})$,
  - $f_{j,i} \colon X_i \to X_j$ continuous for all $j < i$, and
  - $f_{k,i} = f_{k,j} \circ f_{j,i}$ for all $k < j < i$.
- Given a poset diagram $(\vec{X}, \vec{f})$ and $I \subseteq \mathrm{dom}(\vec{X})$, let

$$\lim(X_i : i \in I) = \left\{ p \in \prod_{i \in I} X_i : \forall \{j < i\} \subseteq I \ \ p(j) = f_{j,i}(p(i)) \right\}.$$

- Call a poset diagram $(\vec{X}, \vec{f})$ *multicommutative* if, for all $i \in \mathrm{dom}(\vec{X})$, $\prod_{j<i} f_{j,i}$ maps X_i onto $\lim(X_j : j < i)$.

A new characterization of AE(0)

- A poset P is called *locally finite* if every lower cone is finite.
  - A poset diagram $(\vec{X}, \vec{f})$ is *locally finite* if $\mathrm{dom}(\vec{X})$ is locally finite **and every X_i is finite**.
- A locally finite poset is a *lattice* iff every nonempty finite subset has a least upper bound.
  - A poset diagram $(\vec{X}, \vec{f})$ is called a *lattice diagram* if $\mathrm{dom}(\vec{X})$ is a lattice.

Theorem

Given a boolean space X, the following are equivalent.

- *X is AE(0).*
- *X is homeomorphic to the limit of a multicommutative locally finite poset diagram.*
- *X is homeomorphic to the limit of a multicommutative locally finite lattice diagram.*

Long ω_1-approximation sequences

- For every ordinal α, let
  - $\lfloor \alpha \rfloor = \max\{\beta \leq \alpha : \beta < \omega_1 \text{ or } \exists \gamma \ \ |\alpha| \cdot \gamma = \beta\}$;
  - $\alpha = \lfloor \alpha \rfloor + [\alpha]$;

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  - $\lfloor\alpha\rfloor_n = \sum_{i<n} \lfloor[\alpha]_i\rfloor$;
  - $\daleth(\alpha) = \min\{n < \omega : [\alpha]_n = 0\}$.
  - If $1 \leq k < \omega$ and $\alpha \leq \omega_k$, then $\daleth(\alpha) \leq k$.
- Given θ regular and uncountable, a *long ω_1-approximation sequence* is a transfinite sequence $(M_\alpha)_{\alpha<\eta}$ of countable elementary substructures of $H(\theta)$ such that $(M_\beta)_{\beta<\alpha} \in M_\alpha$ for all $\alpha < \eta$.
- (Milovich, 2008) If $\vec{M}$ is a long ω_1-approximation sequence and $\alpha, \beta \in \mathrm{dom}(\vec{M})$, then
  - $M_\beta \in M_\alpha \Leftrightarrow \beta \in \alpha \cap M_\alpha \Leftrightarrow M_\beta \subsetneq M_\alpha$;
  - for all $i < \daleth(\alpha)$, $M_\alpha^i = \bigcup\{M_\gamma : \lfloor\alpha\rfloor_i \leq \gamma < \lfloor\alpha\rfloor_{i+1}\}$ is a directed union; hence, $M_\alpha^i \prec H(\theta)$.

n-commutativity

- Call a lattice diagram $(\vec{X}, \vec{f})$ *n-commutative* if, for all $i \in \mathrm{dom}(\vec{X})$ and all $j_0, \ldots, j_{n-1} < i$, $\prod_{k \in K} f_{k,i}$ maps X_i onto $\lim(X_k : k \in K)$ where $K = \bigcup_{m<n}\{k : k \leq j_m\}$.
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- Call a boolean space *n-commutative* if it is homeomorphic to the limit of an n-commutative locally finite lattice diagram.
- The Stone dual of "2-commutative boolean space" has been studied under the name of "*strong Freese-Nation property*."
- There are 2-commutative boolean spaces of weight $\aleph_2$ that are known to not be AE(0), *e.g.*, the symmetric square of 2^{ω_2}.

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- There are 2-commutative boolean spaces of weight $\aleph_2$ that are known to not be AE(0), *e.g.*, the symmetric square of 2^{ω_2}.
- (Milovich) Every locally finite lattice of size $\aleph_{n-1}$ contains a cofinal suborder that is an *n*-ladder, *i.e.*, a locally finite lattice in which every element has at most *n* maximal strict lower bounds.
- Hence, a boolean space of weight $\aleph_{n-1}$ is AE(0) iff it is n-commutative.
- Hence, there are 2-commutative boolean spaces of weight $\aleph_2$ that are not 3-commutative.

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- If A is a boolean algebra, $(M_\alpha)_{\alpha<|A|}$ is a long ω_1-approximation sequence, and $A \in M_0$, then, for all $\alpha < |A|$, $i < \daleth(\alpha)$, and $a \in A \cap M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta$, set $\sigma_i(a) = \min\{b \in A \cap M^i_\alpha : b \geq a\}$ if it exists.

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- (Milovich) A has the FN iff, for all $\vec{M}, \alpha, i, a$ as above, $\sigma_i(a)$ exists.
- (Milovich) A has the SFN iff, for all $\vec{M}$ as above, $A' = (A, \wedge, \vee, -, \sigma_0, \sigma_1, \sigma_2, \ldots)$ is a locally finite partial algebra, *i.e.*, every finite subset of A is contained in a finite subalgebra of A'.
- (Milovich) There is a boolean algebra of size $\aleph_2$ that has the FN but not the SFN.