Noetherian types of homogeneous compacta

David Milovich

Spring Topology and Dynamics Conference 2007

Classifying known homogeneous compacta

- **Definition.** A compactum is dyadic if it is a continuous image of a power of 2.

- All known examples of homogeneous compacta are products of dyadic compacta, first-countable compacta, and/or two "exceptional" kinds of homogenous compacta.

- For example, all compact groups are dyadic.

- The first exception is a carefully chosen resolution topology that is homogeneous assuming $\mathsf{MA} + \neg\mathsf{CH}$ and inhomogeneous assuming CH (van Mill, 2003). This space has π-weight ω and character ω_1. Any product X of dyadic compacta and first countable compacta satisfies $\chi(X) \leq \pi(X)$.

- The second exception is a carefully chosen quotient of $(\mathbb{R}/\mathbb{Z}) \times (2^{\omega\cdot\omega}_{\mathrm{lex}})^{\mathfrak{c}}$ which is exceptional by a connectedness argument (M., 2007).

- That's all we've got. So, what do these spaces have in common?

- **Van Douwen's Problem.** All known homogeneous compacta have cellularity at most $\mathfrak{c}$ (*i.e.*, lack a pairwise disjoint open family of size $\mathfrak{c}^+$). It's open (in all models of ZFC) whether this is true of all homogeneous compacta.

- In analogy with this observed upper bound on cellularity, if we consider certain cardinal functions derived from order-theoretic base properties, then we find nontrivial upper bounds for all known homogeneous compacta.

Noetherian cardinal functions

- **Definition.** A family $\mathcal{U}$ of sets if κ^{op}-like if no element of $\mathcal{U}$ has κ-many supersets in $\mathcal{U}$.

- **Definition** (Peregudov, 1997). The *Noetherian type* $Nt(X)$ of a space X is the least κ such that X has a κ^{op}-like base.

- **Definition** (Peregudov, 1997). The *Noetherian* π*-type* $\pi Nt(X)$ of a space X is the least κ such that X has a κ^{op}-like π-base.

- **Definition.** The *local Noetherian type* $\chi Nt(p, X)$ of a point p in a space X is the least κ such that p has a κ^{op}-like local base. Set $\chi Nt(X) = \sup_{p \in X} \chi Nt(p, X)$.

- Every known example of a homogeneous compactum X satisfies

$$\begin{aligned} Nt(X) &\leq \mathfrak{c}^+, \\ \pi Nt(X) &\leq \omega_1, \text{ and} \\ \chi Nt(X) &= \omega. \end{aligned}$$

- **Question.** Are any of these bounds true for all homogeneous compacta?

- Are these bounds sharp? The double arrow space has Noetherian type $\mathfrak{c}^+$ and Suslin lines have Noetherian π-type ω_1.

- **Question.** Is there a ZFC example of a homogeneous compactum with uncountable Noetherian π-type?

Products behave nicely.

- **Theorem** (Peregudov, 1997). $Nt\left(\prod_{i\in I} X_i\right) \leq \sup_{i\in I} Nt\left(X_i\right)$.

  Similarly,

$$\pi Nt\left(\prod_{i\in I} X_i\right) \leq \sup_{i\in I} \pi Nt\left(X_i\right) \text{ and}$$

$$\chi Nt\left(p, \prod_{i\in I} X_i\right) \leq \sup_{i\in I} \chi Nt\left(p(i), X_i\right).$$

- **Theorem** (Malykhin, 1987). Assume X_i is T_1 and $|X_i| \geq 2$ for all $i \in I$. If $|I| \geq \sup_{i\in I} w(X_i)$, then $Nt\left(\prod_{i\in I} X_i\right) = \omega$. In particular, $Nt\left(X^{w(X)}\right) = \omega$ for all T_1 spaces X.

First countable compacta

- **Lemma.** For all spaces X and all points p in X, we have

$$\chi Nt\,(p, X) \le \chi(p, X),$$
$$\pi Nt\,(X) \le \pi(X), \text{ and}$$
$$Nt\,(X) \le w(X)^+.$$

- **Lemma.** For all compacta X, we have

$$\pi Nt\,(X) \le t(X)^+ \le \chi(X)^+.$$

- **Theorem 1.** Let X be a first countable compactum. Then $Nt\,(X) \le \mathfrak{c}^+$ and $\pi Nt\,(X) \le \omega_1$ and $\chi Nt\,(X) = \omega$.

Dyadic compacta

- **Theorem 2.** Let X be a dyadic compactum. Then

$$\chi Nt\,(X) = \pi Nt\,(X) = \omega.$$

- **Theorem 3.** Suppose X is a dyadic compactum and $\pi\chi(p, X) = w(X)$ for all $p \in X$. Then $Nt\,(X) = \omega$.

- **Corollary.** Let X be a homogeneous dyadic compactum. Then $Nt\,(X) = \omega$.

About the proofs of Theorems 2 and 3

- By Stone duality, a dyadic compactum is closely connected to a free boolean algebra. Free boolean algebras have very well-behaved elementary substructures.

- We construct the relevant ω^{op}-like families of open sets iteratively, at each stage working with a quotient space $X/\equiv_M$, where M is a sufficiently small elementary substructure of H_θ and $p \equiv_M q$ iff $f(p) = f(q)$ for all continuous $f\colon X \to \mathbb{R}$ in M.

- For Theorem 2, we use an elementary chain of substructures of H_θ. For Theorem 3, we use a carefully arranged tree of substructures of H_θ (Jackson and Mauldin, 2002).

More about $\chi Nt\,(\cdot)$

- **Theorem 4**. Let X be a compactum. If $\pi\chi(p, X) = \chi(X)$ for all $p \in X$, then $\chi Nt\,(p, X) = \omega$ for some $p \in X$.

- **Corollary** (GCH). For all homogeneous compacta X, we have

$$\chi Nt\,(X) \leq c(X).$$

- **Theorem 5**. Suppose X is a compactum, $\chi(X) = 2^{\kappa}$, and $u(\kappa)$, the space of uniform ultrafilters on κ, embeds into X. Then $\chi Nt\,(p, X) = \omega$ for some $p \in X$.

References

V. I. Malykhin, *On Noetherian Spaces*, Amer. Math. Soc. Transl. **134** (1987), 83–91.

D. Milovich, *Amalgams, connectifications, and homogeneous compacta*, Topology and its Applications **154** (2007), 1170–1177.

S. A. Peregudov, *On the Noetherian type of topological spaces*, Comment. Math. Univ. Carolin. **38** (1997), 581–586.

J. van Mill, *On the character and π-weight of homogeneous compacta*, Israel J. Math. **133** (2003), 321–338.