Diamond and ultrafilters

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Tukey equivalence

- **Definition/Fact.** A directed set $P$ is Tukey reducible to a directed set $Q$ (written $P \leq_T Q$) if and only if one of the following equivalent statements holds.

  - There is a map from $P$ to $Q$ such that the image of every unbounded set is unbounded.

  - There is a map from $P$ to $Q$ such that the preimage of every bounded set is bounded.

  - There is a map from $Q$ to $P$ such that the image of every cofinal subset is cofinal.
• If $P \leq_T Q \leq_T P$, then we say $P$ and $Q$ are Tukey equivalent, writing $P \equiv_T Q$.

• **Theorem** (Tukey, 1940). $P \equiv_T Q$ iff $P$ and $Q$ order-embed as cofinal subsets of a common third directed set.

• Every countable directed set is Tukey-equivalent to 1 (the singleton order) or $\omega$ (an ascending sequence).

• The $\omega_1$-sized directed sets are Tukey equivalent to 1, $\omega$, $\omega_1$, $\omega \times \omega_1$ (with the product order), $[\omega_1]^{<\omega}$ (the finite subsets of $\omega_1$ ordered by inclusion), or maybe something else. (E.g., PFA implies these five are exhaustive; CH implies there are $2^{\omega_1}$ more possibilities (Todorčević, 1985).)
What’s this got to do with topology?

- **Convention.** Families of open sets are ordered by $\supseteq$.

- **Theorem.** Suppose $X$ and $Y$ are spaces, $p \in X$, $q \in Y$, $\mathcal{A}$ is a local base at $p$ in $X$, $\mathcal{B}$ is a local base at $q$ in $Y$, $f : X \to Y$ is continuous and open (or just continuous at $p$ and open at $p$), and $f(p) = q$. Then $\mathcal{B} \leq_T \mathcal{A}$.

- **Proof.** Choose $H : \mathcal{A} \to \mathcal{B}$ such that $H(U) \subseteq f[U]$ for all $U \in \mathcal{A}$. (Here we use that $f$ is open.) Suppose $\mathcal{C} \subseteq \mathcal{A}$ is cofinal. For any $U \in \mathcal{B}$, we may choose $V \in \mathcal{A}$ such that $f[V] \subseteq U$ by continuity of $f$. Then choose $W \in \mathcal{C}$ such that $W \subseteq V$. Hence, $H(W) \subseteq f[W] \subseteq f[V] \subseteq U$. Thus, $H[\mathcal{C}]$ is cofinal.
• **Corollary.** In the above theorem, if $f$ is a homeomorphism, then every local base at $p$ is Tukey-equivalent to every local base at $q$.

• Thus, the Tukey class of a point’s local bases is a topological invariant.
For example, consider the ordered space \( X = \omega_1 + 1 + \omega^* \). It has a point \( p \) that is the limit of an ascending \( \omega_1 \)-sequence and a descending \( \omega \)-sequence. Every local base at \( p \) (when ordered by \( \supseteq \)) is Tukey equivalent to the product order \( \omega \times \omega_1 \).

Next, consider \( D_{\omega_1} \cup \{\infty\} \), the one-point compactification of the \( \omega_1 \)-sized discrete space. Glue \( X \) and \( D_{\omega_1} \cup \{\infty\} \) together into a new space \( Y \) by a quotient map that identifies \( p \) and \( \infty \). Think of \( Y \) as \( X \) with a cloud of points attached to \( p \). In \( Y \), every local base at \( p \) is Tukey equivalent to \([\omega_1]^{<\omega}\) (the finite subsets of \( \omega_1 \) ordered by inclusion), which is not Tukey equivalent to \( \omega \times \omega_1 \).

Thus, we can distinguish \( p \) in \( X \) from \( p \) in \( Y \) by their associated Tukey classes, even though other topological properties, such as character and \( \pi \)-character, have not changed.
The spaces $\beta\omega$ and $\beta\omega \setminus \omega$

- By Stone duality, every ultrafilter $\mathcal{U}$ on $\omega$ is such that $\mathcal{U}$ ordered by $\supseteq$ is Tukey-equivalent to every local base of $\mathcal{U}$ in $\beta\omega$.

- Likewise, $\mathcal{U}$ ordered by $\supseteq^*$ (containment mod finite) is Tukey equivalent to every local base of $\mathcal{U}$ in $\beta\omega \setminus \omega$.

- Thus, the classification the Tukey classes of local bases in $\beta\omega$ and $\beta\omega \setminus \omega$ reduces to a problem of infinite combinatorics.
• **Theorem** (Isbell, 1965). There exists \( \mathcal{U} \in \beta \omega \setminus \omega \) such that \( \langle \mathcal{U}, \supseteq \rangle \equiv_T \langle \mathcal{U}, \supseteq^* \rangle \equiv_T [c]^\omega \) (the finite sets of reals ordered by inclusion).

• Every directed set \( \mathcal{Q} \) of size at most \( c \) satisfies \( 1 \leq_T \mathcal{Q} \leq_T [c]^\omega \), so 1 and \([c]^\omega\) are the minimum and maximum Tukey classes among ultrafilters on \( \omega \), whether ordered by \( \supseteq \) or \( \supseteq^* \).

• Every principal ultrafilter is trivially Tukey equivalent to 1.

• **Question** (Isbell, 1965). Is there a \( \mathcal{U} \in \beta \omega \) such that

\[
1 <_T \langle \mathcal{U}, \supseteq \rangle <_T [c]^\omega ?
\]
Don’t take the easy way out.

• For all $\mathcal{U} \in \beta\omega \setminus \omega$, we have $\langle \mathcal{U}, \supseteq^* \rangle \leq_T \langle \mathcal{U}, \supseteq \rangle$. (Proof: use the identity map.)

• If $\mu < \mathfrak{c}$, that is, if some $\mathcal{U} \in \beta\omega \setminus \omega$ has character $\kappa < \mathfrak{c}$, then a trivial cardinality argument shows that

$$1 <_T \langle \mathcal{U}, \supseteq^* \rangle \leq_T \langle \mathcal{U}, \supseteq \rangle \leq_T [\kappa]^{<\omega} <_T [\mathfrak{c}]^{<\omega}.$$ 

• It’s easy to force $\mu < \mathfrak{c}$.

• To make things interesting, we’ll restrict our attention to $\mathcal{U} \in \beta\omega \setminus \omega$ with character $\mathfrak{c}$. We’ll call the Tukey classes of $\langle \mathcal{U}, \supseteq \rangle$ and $\langle \mathcal{U}, \supseteq^* \rangle$ for such $\mathcal{U}$ “big” Tukey classes.
Certain Tukey classes just can’t occur among local bases in \( \beta \omega \) or \( \beta \omega \setminus \omega \). Most of the ones below are ruled out by simple cardinality arguments.

**Theorem.** Suppose \( U \in \beta \omega \setminus \omega \). Then \( \langle U, \supseteq \rangle \) is not Tukey equivalent to 1, \( \omega \), \( \omega_1 \), \( \omega \times \omega_1 \), or to any countable union of \( \sigma \)-directed sets. Moreover, \( \langle U, \supseteq^* \rangle \) is not Tukey equivalent to any of 1, \( \omega \), \( \omega \times \omega_1 \), or \( \omega \times Q \) where \( Q \) is any countable union of \( \sigma \)-directed sets.

On the other hand, CH implies there exists \( U \in \beta \omega \setminus \omega \) such that \( \omega_1 \equiv_T \langle U, \supseteq^* \rangle <_T \langle U, \supseteq \rangle \).

Note that if \( U \in \beta \omega \setminus \omega \), then by definition \( \langle U, \supseteq^* \rangle \) is \( \sigma \)-directed if and only if \( U \) is a P-point in \( \beta \omega \setminus \omega \).
Main Theorem. Assuming ♦, there exists $U \in \beta\omega \setminus \omega$ such that $U$ has character $\mathfrak{c}$ and $1 <_{T} \langle U, \supseteq^* \rangle \leq_{T} \langle U, \supseteq \rangle <_{T} [\mathfrak{c}]^{<\omega}$. Thus, Isbell’s question consistently has a positive answer even when restricted to big Tukey classes.

♦ can be weakened to $\text{MA}_{\sigma}$-centered + ♦($S^\mathfrak{c}_\omega$) where

$$S^\mathfrak{c}_\omega = \{ \alpha < \mathfrak{c} : \text{cf} \alpha = \omega \}.$$ 

Question. Can ♦ be weakened to CH? Even a ZFC proof has yet to be ruled out.
About the proof

- For all $U \in \beta \omega \setminus \omega$, $\langle U, \supseteq \rangle <_T [\mathfrak{c}]^\omega$ is equivalent to a purely combinatorial statement:

$$\forall A \in [U]^\mathfrak{c} \exists B \in [A]^\omega \bigcap B \in U.$$ 

(For the weaker $\langle U, \supseteq^* \rangle <_T [\mathfrak{c}]^\omega$, one only needs $B$ to have a pseudointersection in $U$.)

- Using ♦ to diagonalize against all $\mathfrak{c}$-sized subsets of $U$, we can construct $U \in \beta \omega \setminus \omega$ such that $U$ is not a P-point and $U$ has character $\mathfrak{c}$ and we have that $\forall A \in [U]^\mathfrak{c} \exists B \in [A]^\omega \bigcap B \in U$. 


Why bother to ensure \( \mathcal{U} \) is not a P-point? Because it hasn’t been done before. Any P-point \( \mathcal{V} \) already satisfies \( \langle \mathcal{V}, \supseteq^* \rangle <_T [\mathfrak{c}]^{< \omega} \). To have a non-P-point \( \mathcal{U} \) satisfying \( \langle \mathcal{U}, \supseteq^* \rangle <_T [\mathfrak{c}]^{< \omega} \) is new.

More generally, forcing gives us relative freedom in constructing P-points of various Tukey classes. For example, there is a ccc order that forces \( \mathfrak{c} = \omega_{42} \) and adds a P-point \( \mathcal{V} \) such that \( \langle \mathcal{V}, \supseteq^* \rangle \equiv_T \omega_1 \times \omega_{42} \) (Brendle and Shelah, 1999). For non-P-points, equally powerful techniques are yet to be found.
Some questions

• ♦ implies there are at least three Tukey classes of local bases in $\beta\omega$. Does it imply there are four? infinitely many?

• Is it consistent that there are only two Tukey classes of local bases in $\beta\omega$?

• Is it consistent that there is only one Tukey class of local bases in $\beta\omega \setminus \omega$?

• More ambitiously, is there a model of ZFC with a nice characterization of the Tukey classes of local bases in $\beta\omega$? in $\beta\omega \setminus \omega$?
References


