

Two spectra of Noetherian types

David Milovich
milovich@math.wisc.edu

March 8, 2009
Spring Topology and Dynamics Conference
Gainesville, FL

An order-theoretic weight

Definition

- ▶ A **base** of a space X is a set \mathcal{B} of open subsets of X such that for every open subset U of X and every $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
- ▶ The **weight** $w(X)$ of a space X is the least infinite κ such that X has a base of size at most κ .

Definition (Peregudov)

- ▶ A family of sets \mathcal{F} is κ^{op} -**like** if every set in \mathcal{F} has fewer than κ -many supersets in \mathcal{F} .
- ▶ The **Noetherian type** $Nt(X)$ of a space X is the least infinite κ such that X has a κ^{op} -like base.

Easy examples

Theorem

$Nt(X) \leq w(X)^+$ for every space X .

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$Nt(X) = \omega$ for every compact metric space X .

Proof.

For each $n < \omega$, let \mathcal{U}_n be a finite cover of X by balls of radius 2^{-n} . Then $\bigcup_{n < \omega} \mathcal{U}_n$ is an ω^{op} -like base of X . □

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Theorem

$Nt(2^\kappa) = \omega$ for every κ .

Proof.

For each $\sigma \in \text{Fn}(\kappa, 2)$, set $U_\sigma = \{f \in 2^\kappa : \sigma \subseteq f\}$. Then $U_\sigma \supseteq U_\tau$ iff $\sigma \subseteq \tau$. Hence, $\{U_\sigma : \sigma \in \text{Fn}(\kappa, 2)\}$ is an ω^{op} -like base of 2^κ . □

Products

Theorem

If $X = \prod_{i \in I} X_i$ and each X_i has a nontrivial open subset, then $w(X) = \sum_{i \in I} w(X_i)$.

Theorem (Peregudov)

If $X = \prod_{i \in I} X_i$, then $Nt(X) \leq \sup_{i \in I} Nt(X_i)$.

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Theorem (Malykhin)

If $X = \prod_{i \in I} X_i$, $w(X) \leq |I|$, and each X_i is a nontrivial union of two open sets, then $Nt(X) = \omega$.

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If $X = \prod_{i \in I} X_i$, $w(X) \leq |I|$, and each X_i is a nontrivial union of two open sets, then $Nt(X) = \omega$.

Question

Are there a spaces X, Y such that $Nt(X \times Y) < Nt(X)Nt(Y)$?

$\beta\mathbb{N} \setminus \mathbb{N}$

Theorem (M.)

Each of the following is consistent with ZFC.

- ▶ $Nt(\beta\mathbb{N} \setminus \mathbb{N}) = \omega_1 = 2^\omega$ (Malykhin)
- ▶ $\omega_1 < Nt(\beta\mathbb{N} \setminus \mathbb{N}) = 2^\omega$
- ▶ $Nt(\beta\mathbb{N} \setminus \mathbb{N}) = \omega_1 < 2^\omega$
- ▶ $\omega_1 < Nt(\beta\mathbb{N} \setminus \mathbb{N}) < 2^\omega$
- ▶ $\omega_1 < 2^\omega < (2^\omega)^+ = Nt(\beta\mathbb{N} \setminus \mathbb{N})$

Much more can be said in terms of \mathfrak{s} , \mathfrak{r} , \mathfrak{b} , \mathfrak{d} , \mathfrak{u} , \mathfrak{h} , \mathfrak{i} , and \mathfrak{p} . (E.g., $Nt(\beta\mathbb{N} \setminus \mathbb{N}) \geq \mathfrak{s}$ and $\text{Con}(\mathfrak{b} < Nt(\beta\mathbb{N} \setminus \mathbb{N}) < \mathfrak{d})$.)

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Much more can be said in terms of \mathfrak{s} , \mathfrak{r} , \mathfrak{b} , \mathfrak{d} , \mathfrak{u} , \mathfrak{h} , \mathfrak{i} , and \mathfrak{p} . (E.g., $Nt(\beta\mathbb{N} \setminus \mathbb{N}) \geq \mathfrak{s}$ and $\text{Con}(\mathfrak{b} < Nt(\beta\mathbb{N} \setminus \mathbb{N}) < \mathfrak{d})$.)

Theorem (M.)

Suppose $|I| < 2^\omega$ and $w(X_i) \leq 2^\omega$ for all $i \in I$. Then

$\prod_{i \in I} (X_i \oplus (\beta\mathbb{N} \setminus \mathbb{N}))$ is not homeomorphic to a product of 2^ω -many non-singleton spaces.

Van Douwen's Problem

Definition

- ▶ A **compactum** is a compact Hausdorff space.
- ▶ A space is **homogeneous** if for every $p, q \in X$ there is an homeomorphism $h: X \rightarrow X$ such that $h(p) = q$.
- ▶ The **cellularity** $c(X)$ of a space X is the least infinite κ such that every pairwise disjoint open family in X has size at most κ .

Question (Van Douwen)

Is there a homogeneous compactum X such that $c(X) > 2^\omega$?

After over 30 years, Van Douwen's Problem has not been solved in any model of ZFC.

The difficulty is structural

Fact

Every known example of a homogeneous compactum X is a continuous image of a product of compacta each with weight at most 2^ω . Hence, $c(X) \leq 2^\omega$. (The upper bound is attained.)

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Theorem A (M.)

If X is a homogeneous compactum and a continuous image of a product $\prod_{i \in I} X_i$ of compacta such that

$$\sup_{i \in I} w(X_i) = \kappa < \text{cf } \lambda = \lambda \leq w(X),$$

then $Nt(X) \leq \kappa$.

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If X is a homogeneous compactum and a continuous image of a product $\prod_{i \in I} X_i$ of compacta such that

$$\sup_{i \in I} w(X_i) = \kappa < \text{cf } \lambda = \lambda \leq w(X),$$

then $Nt(X) \leq \kappa$.

Corollary

Every known homogeneous compactum X satisfies $Nt(X) \leq (2^\omega)^+$. (The upper bound is attained.)

Dyadic compacta

Definition

A **dyadic compactum** is a Hausdorff space that is a continuous image of 2^κ for some κ .

Corollary

$Nt(X) = \omega$ for every homogeneous dyadic compactum X .

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Theorem (M.)

Let $\kappa < \lambda$ be infinite cardinals and let X be the quotient of $2^\kappa \oplus 2^\lambda$ obtained by identifying $\langle 0 \rangle_{i < \kappa}$ and $\langle 0 \rangle_{i < \lambda}$. If $\kappa < \text{cf } \lambda$, then $Nt(X) = \lambda^+$. If $\kappa = \text{cf } \lambda$, then $Nt(X) = \lambda$.

Corollary

The class of Noetherian types of dyadic compacta includes all infinite cardinals except possibly weak inaccessible cardinals and successors of cardinals with countable cofinality (like ω_1 and $\omega_{\omega+1}$).

Non-triviality

Theorem (M.)

The class of Noetherian types of compacta includes all infinite cardinals.

Theorem B (M.)

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Question

Are $\omega_{\omega+1}$ and weak inaccessible cardinals excluded too?

Another spectrum

Perhaps things are easier with ordered compacta.

Theorem (M.)

With respect to the order topology, $Nt(\kappa + 1) = \kappa^+$ if κ is a regular uncountable cardinal and $Nt(\kappa + 1) = \kappa$ if κ is a singular cardinal.

Proof.

By the Pressing Down Lemma...



Excluding ω_1

Theorem C (M.)

If X is an ordered compactum and $Nt(X) \leq \kappa$ and κ is regular and uncountable, then X has a dense set of size less than κ .

Corollary

If X is an ordered compactum, then $Nt(X) \neq \omega_1$.

Excluding ω_1

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Corollary

If X is an ordered compactum, then $Nt(X) \neq \omega_1$.

Proof.

Suppose $Nt(X) = \omega_1$. Then X has a countable dense subset D . Also, X is not metrizable, so $w(X) \geq \omega_1$. Let \mathcal{B} be a base of X . Then for some $p, q \in D$ and $U \in \mathcal{B}$ we have $U \subseteq (p, q) \subseteq V$ for uncountably many $V \in \mathcal{B}$. Thus, $Nt(X) \geq \omega_2$. □

Excluding weak inaccessible

Corollary

If X is an ordered compactum, then $Nt(X)$ is not a weak inaccessible.

Proof.

Suppose $\kappa = Nt(X)$ is weakly inaccessible. Then X has a dense subsets D of size less than κ . If $w(X) \geq \kappa$, then, arguing as before, $Nt(X) \geq \kappa^+$. If $w(X) < \kappa$, then $Nt(X) \leq w(X)^+ < \kappa$. \square

A spectrum characterized

Theorem D (M.)

For each singular cardinal κ , there is an ordered compactum X such that $Nt(X) = \kappa^+$.

Corollary

The class of Noetherian types of ordered compacta includes all infinite cardinals except ω_1 and the weak inaccessibles.

Questions

Question

Do the dyadic compacta have the same Noetherian spectrum as the ordered compacta?

Question

Is there an interesting example of a class of spaces with Noetherian spectrum excluding ω_2 ?

Proving Theorems A and B

Theorem A

If X is a homogeneous compactum and a continuous image of a product $\prod_{i \in I} X_i$ of compacta such that

$$\sup_{i \in I} w(X_i) = \kappa < \text{cf } \lambda = \lambda \leq w(X),$$

then $Nt(X) \leq \kappa$.

Theorem B

The class of Noetherian types of dyadic compacta excludes ω_1 .

Substructures and quotients

Definition

- ▶ $H(\theta)$ is the set of sets that are hereditarily smaller than θ .
- ▶ $C(X)$ is the set of continuous maps from X to \mathbb{R} .
- ▶ Given X a compactum, θ a sufficiently large regular cardinal, and M an elementary substructure of $\langle H(\theta), \in, <, C(X) \rangle$, define a quotient map $\pi_M^X: X \rightarrow X/M$ by

$$\pi_M^X(p) \neq \pi_M^X(q) \text{ iff } f(p) \neq f(q) \text{ for some } f \in C(X) \cap M.$$

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- ▶ Compact metric spaces have very nice ω^{op} -like bases.
- ▶ We may choose a sequence $\langle M_\alpha \rangle_{\alpha < w(X)}$ of countable $M_\alpha \prec H(\theta)$ such that $\alpha \in M_\alpha$ for all $\alpha < w(X)$.

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- ▶ Given a base \mathcal{B}_α of each X/M_α , $\mathcal{B} = \bigcup_{\alpha < w(X)} (\pi_{M_\alpha}^X)^{-1}(\mathcal{B}_\alpha)$ is a base of X .

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- ▶ Given a base \mathcal{B}_α of each X/M_α , $\mathcal{B} = \bigcup_{\alpha < w(X)} (\pi_{M_\alpha}^X)^{-1}(\mathcal{B}_\alpha)$ is a base of X .
- ▶ If the substructures cohere sufficiently well, then we can use reflection arguments (and $\min_{p \in X} \pi_\chi(p, X) = w(X)$) to carefully construct subsets $\mathcal{A}_\alpha \subseteq \mathcal{B}_\alpha$ such that $\mathcal{A} = \bigcup_{\alpha < w(X)} (\pi_{M_\alpha}^X)^{-1}(\mathcal{A}_\alpha)$ is an ω^{op} -like base of X .

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Let Ω denote the class of finite, nonempty ordinal sequences $\langle \gamma_i \rangle_{i < n}$ for which $|\gamma_0| > |\gamma_1| > \cdots > |\gamma_{n-2}| \geq \omega_1 > |\gamma_{n-1}|$ if $n \geq 2$ and $\omega_1 > |\gamma_{n-1}|$ if $n = 1$.

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Let \sqsubseteq denote the lexicographic ordering of Ω . Since \sqsubseteq is a well-ordering, there is a unique isomorphism Υ from the ordinals to $\langle \Omega, \sqsubseteq \rangle$.

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Let \sqsubseteq denote the lexicographic ordering of Ω . Since \sqsubseteq is a well-ordering, there is a unique isomorphism Υ from the ordinals to $\langle \Omega, \sqsubseteq \rangle$. For each $\Upsilon(\alpha) = \langle \gamma_i \rangle_{i < n}$ and $k < n - 1$, define:

$$N_{\alpha,k} = \bigcup \{ M_\beta : \langle \gamma_0, \dots, \gamma_{k-1}, 0 \rangle \sqsubseteq \Upsilon(\beta) \sqsubset \langle \gamma_0, \dots, \gamma_k, 0 \rangle \}$$

$$N_{\alpha,n-1} = \bigcup \{ M_\beta : \langle \gamma_0, \dots, \gamma_{n-2}, 0 \rangle \sqsubseteq \Upsilon(\beta) \sqsubset \langle \gamma_0, \dots, \gamma_{n-1} \rangle \}$$

Coherence as promised

Theorem

- ▶ $\bigcup_{i < n} N_{\alpha,i} = \bigcup_{\beta < \alpha} M_{\beta}$
- ▶ $N_{\alpha,i} \in M_{\alpha}$ for all $i < n$.
- ▶ $N_{\alpha,i} \prec H(\theta)$ for all $i < n$.

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Jackson and Mauldin first constructed a tree of substructures satisfying the above theorem. I just showed that one can build the tree from a mere sequence of M_α 's satisfying $\langle M_\beta \rangle_{\beta < \alpha} \in M_\alpha$.

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Remark

If $w(X) = \omega_1$, then we don't need such fancy machinery; a continuous elementary chain of countable submodels suffices.

Proving Theorem C

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Proof.

- ▶ Let \mathcal{B} be κ^{op} -like base of X . Let $M \prec \langle H(\theta), \in, \mathcal{B} \rangle$, $|M| < \kappa$, and $M \cap [H(\theta)]^{<\kappa} \subseteq [M]^{<\kappa}$.

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- ▶ If $p, q \in X \cap M$ and $\emptyset \neq (p, q) \subseteq U \in \mathcal{B}$, then $U \in M$, so points like $\min\{x \in X : q \leq x \notin U\}$ are also in M .

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- ▶ If $p, q \in X \cap M$ and $\emptyset \neq (p, q) \subseteq U \in \mathcal{B}$, then $U \in M$, so points like $\min\{x \in X : q \leq x \notin U\}$ are also in M .
- ▶ It follows that $X \cap M$ is dense in X .

□

Proving Theorem D

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- ▶ Let $\lambda = \text{cf } \kappa$ and $Y = \lambda^+ + 1$.

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- ▶ Let $\lambda = \text{cf } \kappa$ and $Y = \lambda^+ + 1$.
- ▶ Partition the limit ordinals less than λ^+ into stationary sets $\langle S_\alpha \rangle_{\alpha < \lambda}$.

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Proof.

- ▶ Let $\lambda = \text{cf } \kappa$ and $Y = \lambda^+ + 1$.
- ▶ Partition the limit ordinals less than λ^+ into stationary sets $\langle S_\alpha \rangle_{\alpha < \lambda}$.
- ▶ Let $\langle \kappa_\alpha \rangle_{\alpha < \lambda}$ be an increasing sequence of regular cardinals with supremum κ .

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- ▶ Let $\langle \kappa_\alpha \rangle_{\alpha < \lambda}$ be an increasing sequence of regular cardinals with supremum κ .
- ▶ For each $\alpha < \lambda$ and $\beta \in S_\alpha$, set $Z_\beta = (\kappa_\alpha + 1)^{\text{op}}$.

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- ▶ For each $\alpha < \lambda$ and $\beta \in S_\alpha$, set $Z_\beta = (\kappa_\alpha + 1)^{\text{op}}$.
- ▶ For each $\beta \in Y \setminus \bigcup_{\alpha < \lambda} S_\alpha$, set $Z_\beta = 1$.

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- ▶ For each $\beta \in Y \setminus \bigcup_{\alpha < \lambda} S_\alpha$, set $Z_\beta = 1$.
- ▶ $X = \sum_{\beta \in Y} Z_\beta$.

