On the order theory of local bases

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Order theory Preliminaries

Definition

- A preorder $P$ is $\kappa$-directed if every subset smaller than $\kappa$ has an (upper) bound in $P$.

- Directed means $\aleph_0$-directed.
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Conversely:

▶ A preorder $P$ is $\kappa$-short if every bounded subset is smaller than $\kappa$.

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A preorder $P$ is **almost $\kappa$-short** if it has a $\kappa$-short cofinal suborder.
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Definition

A preorder $P$ is almost $\kappa$-short if it has a $\kappa$-short cofinal suborder.

Convention

Order sets like $[\lambda]^\kappa$ and $2^{<\kappa}$ by $\subseteq$. 
Classifying preorders

**Definition**
Two preorders $P$ and $Q$ are **mutually cofinal** if they are isomorphic to cofinal suborders of a common third preorder $R$.

**Lemma (M., 2005)**
If $P$ and $Q$ are mutually cofinal and $P$ is almost $\kappa$-short, then $Q$ is almost $\kappa$-short.
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Definition (Tukey, 1940)

- $P$ is **Tukey-below** $Q$, or $P \leq_T Q$, if there exists $f : P \leq_T Q$, i.e., $f : P \to Q$ sends unbounded sets to unbounded sets.
- $P \equiv_T Q$ means $P \leq_T Q \leq_T P$. 

Tukey types aren't cofinal types...
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Tukey types aren’t cofinal types...
$2^{<\omega_1} \equiv_T [c]^1$ and $[c]^1$ is flat, but $2^{<\omega_1}$ is not almost flat.
... until we assume directedness

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Definition
A $(\lambda, \kappa)$-blossom in a preorder $P$ is a map $f : \lambda \to P$ such that $f[I]$ is unbounded for all $I \in [\lambda]^\kappa$. 

Theorem (M., 2007)
If $P$ is directed and $\text{cf}(P) \geq \aleph_0$, then (1) $\Rightarrow$ (2) $\iff$ (3):
1. $\text{cf}(P) < \kappa \leq \text{T}_P$.
2. $P$ has a $(\text{cf}(P), \kappa)$-blossom.
3. $P$ is almost $\kappa$-short.
If also $\kappa = \text{cf}(\kappa)$ and $|\text{cf}(P)| < \kappa = \text{cf}(P)$, then (1) $\iff$ (2).
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**Theorem (M., 2007)**
If $P$ is directed and $\text{cf}(P) \geq \aleph_0$, then (1) $\Rightarrow$ (2) $\iff$ (3):

1. $[\text{cf}(P)]^{<\kappa} \leq_T P$.
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**Proof**

- $(1) \Rightarrow (2)$: If $f : [\text{cf}(P)]^{<\kappa} \leq_T P$, then $\langle f(\{\alpha\}) \rangle_{\alpha < \text{cf}(P)}$ is a $(\text{cf}(P), \kappa)$-blossom.
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- $(2) \Rightarrow (3)$: Given a $(\text{cf}(P), \kappa)$-blossom $b$ and $c : \text{cf}(P) \rightarrow P$ with cofinal range, let $d(\alpha) \geq b(\alpha), c(\alpha)$ for all $\alpha$; ran($d$) is cofinal and $\kappa$-short.
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▷ (1) $\Rightarrow$ (2): If $f : [\text{cf}(P)]^{<\kappa} \leq_T P$, then $\langle f(\{\alpha\}) \rangle_{\alpha < \text{cf}(P)}$ is a $(\text{cf}(P), \kappa)$-blossom.

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▷ (2) $\Rightarrow$ (3): Given a $(\text{cf}(P), \kappa)$-blossom $b$ and $c : \text{cf}(P) \rightarrow P$ with cofinal range, let $d(\alpha) \geq b(\alpha), c(\alpha)$ for all $\alpha$; ran$(d)$ is cofinal and $\kappa$-short.

▷ (2) $\Rightarrow$ (1): Given a $(\text{cf}(P), \kappa)$-blossom $b$ and an injective $h : [\text{cf}(P)]^{<\kappa} \rightarrow \text{cf}(P)$, we have $b \circ h : [\text{cf}(P)]^{<\kappa} \leq_T P$. 
Topological preliminaries

Convention

- All spaces are Tychonoff ($T_{3.5}$).
- Families of open sets are ordered by $\supseteq$. 
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Notation

- $\tau(X)$ is the set of open subsets of $X$.
- $\tau^+(X)$ is the set of nonempty open subsets of $X$.
- $\tau(p, X)$ is the set of open neighborhoods of $p$ in $X$. 
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Definition

- A **local base** at $p$ is a cofinal subset of $\tau(p, X)$.
- A **$\pi$-base** is a cofinal subset of $\tau^+(X)$.
- A **base** is a subset $\mathcal{B}$ of $\tau(X)$ that includes a local base at every point.
<table>
<thead>
<tr>
<th><strong>The weight</strong> $w(X)$ of $X$ is the least $\kappa \geq \aleph_0$ such that $X$ has a base that is of size $\leq \kappa$.</th>
<th><strong>The Noetherian type</strong> $\text{Nt}(X)$ of $X$ is the least $\kappa \geq \aleph_0$ such that $X$ has a base that is $\kappa$-short.</th>
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<tr>
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<td><strong>The local Noetherian type</strong> $\chi\text{Nt}(p, X)$ of $p$ in $X$ is the least $\kappa \geq \aleph_0$ such that $p$ has a local base that is $\kappa$-short.</td>
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<td>$\chi(X) = \sup_{p \in X} \chi(p, X)$</td>
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History

- Malykhin, Peregudov, and Šapirovskii studied the properties $N_t(X) \leq \aleph_1$, $\pi N_t(X) \leq \aleph_1$, $N_t(X) = \aleph_0$, and $\pi N_t(X) = \aleph_0$ in the 1970s and 1980s.
- Peregudov introduced Noetherian type and Noetherian $\pi$-type in 1997.
- Milovich introduced local Noetherian type in 2005.
History

- Malykhin, Peregudov, and Šapirovič studied the properties \( Nt(X) \leq \aleph_1, \pi Nt(X) \leq \aleph_1, Nt(X) = \aleph_0, \) and \( \pi Nt(X) = \aleph_0 \) in the 1970s and 1980s.
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Order-theoretic definitions

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<th>Symbol</th>
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Easy upper bounds

**Lemma**
Every preorder $P$ is almost $\text{cf}(P)$-short.

**Corollary**
For all spaces $X$,

- $\chi_{\text{Nt}}(p, X) \leq \chi(p, X)$;
- $\chi_{\text{Nt}}(X) \leq \chi(X)$;
- $\pi_{\text{Nt}}(X) \leq \pi(X)$.

Even easier:
Every $P$ is $|P|$-short, so $\text{Nt}(X) \leq w(X) +$.
Easy upper bounds

**Lemma**
Every preorder $P$ is almost $\text{cf}(P)$-short.

**Corollary**
For all spaces $X$,

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Every $P$ is $|P|^+$-short, so $\mathbb{Nt}(X) \leq w(X)^+$. 
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Every preorder $P$ is almost $\text{cf}(P)$-short.

Corollary
For all spaces $X$,

1. $\chi_{Nt}(p, X) \leq \chi(p, X)$;
2. $\chi_{Nt}(X) \leq \chi(X)$;
3. $\pi_{Nt}(X) \leq \pi(X)$.

Even easier:
Every $P$ is $|P|^+$-short, so $Nt(X) \leq w(X)^+$.

Example
$Nt(\beta\mathbb{N}) = w(\beta\mathbb{N})^+ = c^+$ because $\pi(\beta\mathbb{N}) = \aleph_0 < \text{cf}(w(\beta\mathbb{N}))$. 
Passing to subsets

Applying mutual cofinality

- If \( B \) is a \( \pi \)-base of \( X \), then \( B \) includes a \( \pi \text{Nt} (X) \)-short \( \pi \)-base of \( X \).
- If \( B \) is a local base at \( p \) in \( X \), then \( B \) includes a \( \chi \text{Nt} (X) \)-short local base at \( p \) in \( X \).
Passing to subsets

Applying mutual cofinality

- If $\mathcal{B}$ is a $\pi$-base of $X$, then $\mathcal{B}$ includes a $\pi\text{Nt} (X)$-short $\pi$-base of $X$.
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Theorem (M., 2007)

Every metrizable space has a flat base.

**Proof:** For each $n < \omega$, pick a locally finite open cover refining the balls of radius $2^{-n}$. Take the union.
Passing to subsets

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- If $B$ is a $\pi$-base of $X$, then $B$ includes a $\pi \mathsf{Nt} (X)$-short $\pi$-base of $X$.
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Every metrizable space has a flat base.
Proof: For each $n < \omega$, pick a locally finite open cover refining the balls of radius $2^{-n}$. Take the union.

Example (M., 2009)
Set $X = \mathbb{Z}^\omega$. Let $B$ be the set of all sets of the form $U_{s,n}$ where $s \in \mathbb{Z}^{<\omega}$, $n < \omega$, and $U_{s,n}$ is the set of all $f \in X$ such that $s \upharpoonright i \subseteq f$ for some $i \in [-n, n]$. $B$ a base of $X$, but $B$ has no flat subcover.
Blossoms and splitters

Applying directedness

If $p \in X$ is not isolated, then $\chi_{Nt}(p, X) \leq \kappa$ if and only if $\tau(p, X)$ has a $(\chi(p, X), \kappa)$-blossom, which is just a $\chi(p, X)$-sequence $\vec{U}$ of neighborhoods of $p$ such that $p \notin \text{int} \bigcap_{\alpha \in I} U_\alpha$ for all $I \in [\chi(p, X)]^\kappa$.
Blossoms and splitters

Applying directedness
If \( p \in X \) is not isolated, then \( \chi_{Nt}(p, X) \leq \kappa \) if and only if \( \tau(p, X) \) has a \((\chi(p, X), \kappa)\)-blossom, which is just a \( \chi(p, X) \)-sequence \( \vec{U} \) of neighborhoods of \( p \) such that \( p \notin \text{int}\bigcap_{\alpha \in I} U_{\alpha} \) for all \( I \in [\chi(p, X)]^\kappa \).

Definition
A \((\lambda, \kappa)\)-splitter of \( X \) is a \( \lambda \)-sequence \( \vec{F} \) of finite open covers of \( X \) such that \( \text{int}\bigcap_{\alpha \in I} U_{\alpha} = \emptyset \) for all \( I \in [\chi(p, X)]^\kappa \) and \( \vec{U} \in \prod_{\alpha \in I} F_{\alpha} \).
Blossoms and splitters

Applying directedness

If \( p \in X \) is not isolated, then \( \chi_{N_t}(p, X) \leq \kappa \) if and only if \( \tau(p, X) \) has a \( (\chi(p, X), \kappa) \)-blossom, which is just a \( \chi(p, X) \)-sequence \( \vec{U} \) of neighborhoods of \( p \) such that \( p \not\in \text{int} \bigcap_{\alpha \in I} U_\alpha \) for all \( I \in [\chi(p, X)]^\kappa \).

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Lemma

If \( X \) has a \( (w(X), \kappa) \)-splitter, then \( N_t(X) \leq \kappa \).

Question (M., 2007)

Does \( N_t(\beta\omega \setminus \omega) \leq \kappa \) imply \( \beta\omega \setminus \omega \) has a \( (c, \kappa) \)-splitter in ZFC? (There can be no counterexamples if \( c \) is regular.)
Theorem
If $X = \prod_{\alpha < \kappa} X_{\alpha}$ and $|X_{\alpha}| > 1$ for all $\alpha < \kappa$, then

- $\kappa \geq \chi(p, X) \Rightarrow \chi Nt(p, X) = \aleph_0$;
- $\kappa \geq \chi(X) \Rightarrow \chi Nt(X) = \aleph_0$;
- $\kappa \geq \pi(X) \Rightarrow \pi Nt(X) = \aleph_0$;
- $\kappa \geq w(X) \Rightarrow Nt(X) = \aleph_0$.
Theorem

If \( X = \prod_{\alpha < \kappa} X_\alpha \) and \( |X_\alpha| > 1 \) for all \( \alpha < \kappa \), then

\[ \kappa \geq \chi(p, X) \Rightarrow \chi Nt(p, X) = \aleph_0; \]
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\[ \kappa \geq w(X) \Rightarrow Nt(X) = \aleph_0. \]

Proof (essentially (Malykhin, 1981))

First claim: For each \( \alpha < \chi(p, X) \), choose a nontrivial open neighborhood \( U_\alpha \) of \( p(\alpha) \). Since all open boxes in the product topology have finite support, \( \langle \pi_\alpha^{-1}[U_\alpha] \rangle_{\alpha < \kappa} \) is a \((\chi(p, X), \aleph_0)\)-blossom for \( \tau(p, X) \).
Corollary

\[ N_t(X \times 2^{w(X)}) = \aleph_0. \] (Malykhin, 1981)
Corollary

- $Nt \left( X \times 2^{w(X)} \right) = \aleph_0$. (Malykhin, 1981)
- $\pi Nt \left( X \times 2^{\pi(X)} \right) = \aleph_0$.
- $\chi Nt \left( X \times 2^{\chi(X)} \right) = \aleph_0$. 
Corollary

\[ \mathcal{N}_t (X \times 2^w(X)) = \aleph_0. \] (Malykhin, 1981)

\[ \pi \mathcal{N}_t (X \times 2^{\pi(X)}) = \aleph_0. \]

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Passing to subsets again

Definition
A space $X$ is **homogeneous** if for all $p, q \in X$, there is a bijection $f : X \to X$ with $f(p) = q$ and $f$ and $f^{-1}$ continuous.

Theorem (M., 2009)
Let $\mathcal{B}$ be a base of $X$. $\mathcal{B}$ includes an $\mathcal{Nt}(X)$-short base of $X$ if
- $X$ is metrizable and $X$ is locally compact or $\sigma$-compact,
- $X$ is compact and $\chi(p, X) = w(X)$ for all $p \in X$, or
- $X$ is compact, homogeneous, and $w(X)$ is regular or strong limit.

*About the proof*
For the second case, we build a $(w(X), \kappa)$-splitter consisting of subcovers of an arbitrary base.
For the third case, we use Misˇ cenko's Lemma to deduce that the second case holds or $\mathcal{Nt}(X) = w(X) +$. 
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Van Douwen’s Problem

Definition
The cellularity \( c(X) \) of \( X \) is the least infinite upper bound of the cardinalities of its cellular families, i.e., pairwise disjoint open families.
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Patterns

▶ Every known compact homogeneous space (**CHS**) is a continuous image of a product of compacta with weight at most \( c \).
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The **cellularity** $c(X)$ of $X$ is the least infinite upper bound of the cardinalities of its **cellular families**, i.e., pairwise disjoint open families.

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- Every known compact homogeneous space (**CHS**) is a continuous image of a product of compacta with weight at most $c$.
- It follows that every known CHS has cellularity at most $c$. (Why? Easy: $c^+$ is a caliber of any such space.)
- Van Douwen’s Problem asks whether $c(X) \leq c$ for every CHS $X$. **This is open after ~40 years, in all models of ZFC.**
Van Douwen’s Problem

Definition
The \textbf{cellularity} $c(X)$ of $X$ is the least infinite upper bound of the cardinalities of its \textbf{cellular families}, i.e., pairwise disjoint open families.

Patterns

\begin{itemize}
  \item Every known compact homogeneous space (\textbf{CHS}) is a continuous image of a product of compacta with weight at most $\mathfrak{c}$.
  \item It follows that every known CHS has cellularity at most $\mathfrak{c}$. (Why? Easy: $\mathfrak{c}^+$ is a caliber of any such space.)
  \item Van Douwen’s Problem asks whether $c(X) \leq \mathfrak{c}$ for every CHS $X$. \textbf{This is open after \textasciitilde40 years, in all models of ZFC.}
  \item It also follows that every known CHS has Noetherian type at most $\mathfrak{c}^+$. (Why? Not as easy...)\end{itemize}
Sharp bounds

Example (Maurice, 1964)
The lexicographically ordered space $X = 2_{\text{lex}}^\omega \cdot \omega$ is a CHS satisfying $c(X) = c$.

Example (Peregudov, 1997)
The double-arrow space $X$ is compact, homogeneous, and $\mathcal{Nt}(X) = c^+$. 
Light factors

Theorem (M., 2007)
If $X$ is CHS and a continuous image of a product of compacta all with weight at most $\lambda$, then $\mathcal{N}_t(X) \leq \lambda^+$. If also $\lambda = \aleph_0$ (i.e., $X$ is dyadic), then $\mathcal{N}_t(X) = \aleph_0$. 

Some ideas from the proof
▶ A long $\kappa$-approximation sequence (for regular $\kappa$) is an $\in$-chain $\vec{M}$ of elementary substructures of $H(\theta)$ with $|M_\alpha| \subseteq \kappa \cap M_\alpha \in \kappa \in M_\alpha$ and $\vec{M} \restriction \alpha \in M_\alpha$.
▶ (A. Miller) Generalizing (Jackson, Mauldin, 2002), given $\vec{M}$ as above, there exists $\vec{\Sigma}$ such that $\Sigma_\alpha \in [M_\alpha]_{\aleph_0}$, $\bigcup \Sigma_\alpha = \bigcup (\vec{M} \restriction \alpha)$, and $N \prec H(\theta)$ for all $N \in \Sigma_\alpha$.
▶ The quotient maps $\pi: X \rightarrow X / M_\alpha$ are open.
▶ We can build a $\kappa$-short base of $X$ by taking the union of pullbacks of well-chosen bases of these quotients.
Light factors

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Definition

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About the proof
This time, we don’t need long $\kappa$-approximation sequences. Continuous elementary chains work just fine.
Does every CHS have a flat local base?

Another Pattern

Every known CHS $X$ satisfies $\pi_{Nt}(X) \leq \aleph_1$ and $\chi_{Nt}(X) = \aleph_0$. 
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Attacking Van Douwen’s Problem

- If we found a model of GCH with a CHS $X$ with a local base $B$ such that $B$ is not almost $\aleph_1$-short, then $c(X) > c$.
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- (Arhangel’スキ, 2005) If a product of linear orders is a CHS, then all factors are first countable, and hence have weight at most $c$. 

(Andrzej Nowak, 2007)
\(\pi\)-character is crucial, again.

Assuming GCH, \(\chi_{\text{Nt}}(X) \leq c(X)\) if \(X\) is a CHS.

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- So, assuming GCH, \(\pi\chi(X) < \chi(X)\) implies \(\chi_Nt(X) \leq \chi(X) \leq c(X)\).
More on $\pi$-character

Theorems

- (M., 2008) If $X$ is compact and $\pi_X(p, X) = w(X)$ for all $p \in X$, then $\mathcal{N}_t(X) \leq w(X)$. 

- (M., Spadaro, 2010) If $X$ is compact and $\pi_X(p, X) < w(X)$ for a dense set of points, then $\mathcal{N}_t(X) \geq w(X)$, and $\mathcal{N}_t(X) = w(X) + 1$ if $w(X)$ is regular.

Examples

- (M., 2010) If $X = D_{\aleph_\omega} \cup \{\infty\}$, then $\pi_X(X) = \aleph_0$, $w(X) = \aleph_\omega$, and $\mathcal{N}_t(X) = \aleph_\omega + 1$.

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Definition (Van Douwen)

A space $X$ is **power homogeneous** if $X^\alpha$ is homogeneous for some $\alpha > 0$. 
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- However, it is unknown whether every PHC $X$ satisfies $c(X) \leq c$.
- It is also unknown whether every PHC $X$ has a flat local base.
- Perhaps an easier question: Does GCH imply $\chi_{\mathcal{N}_t}(X) \leq c(X)$ for all PHC $X$?
A partial answer

Definition

$d(X)$ is the least $\kappa \geq \aleph_0$ such that some $D \in [X]^{\leq \kappa}$ is dense in $X$.

Perhaps an even easier question:

Does GCH imply $\chi_{\text{Nt}}(X) \leq d(X)$ for all PHC $X$?

Theorem (M., Ridderbos, 2007)

Given GCH, $X$ PHC, and $\max_{p \in X} \chi(p, X) = \text{cf}(\chi(X)) > d(X)$, there is a nonempty open $U \subseteq X$ such that $\chi_{\text{Nt}}(p, X) = \aleph_0$ for all $p \in U$. 
If we stop worrying about homogeneity...

Sometimes compactness doesn’t matter.

(M., 2009) If $p \in X$ and $\overline{X} = Y$, e.g., $Y = \beta X$, then
$\chi_{\mathcal{N}t}(p, X) = \chi_{\mathcal{N}t}(p, Y)$ and $\pi_{\mathcal{N}t}(X) = \pi_{\mathcal{N}t}(Y)$. On the other hand, $\mathcal{N}t(\mathbb{N}) = \aleph_0$ and $\mathcal{N}t(\beta \mathbb{N}) = c^+$.
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Product spaces can surprise you.

▶ (Todorˇ cevi´ c, 1985) If \( \text{cf}(\kappa) = \kappa = \kappa^{\aleph_0} \), then there exist directed \( P, Q \) with \( P, Q <_T P \times Q \equiv_T [\kappa]^{<\aleph_0} \).

▶ (M., 2010) Using these \( P \) and \( Q \), we can build \( X, Y \) such that
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▶ (Spadaro, 2008) There are compact \( K, L \) with \( Nt(K) = \aleph_2 \), \( Nt(L) = \aleph_3 \), and \( Nt(K \times L) = \aleph_1 \).

Open: Is \( Nt(X^2) \neq Nt(X) \) possible?
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- (Todorčević, 1985) If $\text{cf}(\kappa) = \kappa = \kappa^{\mathfrak{n}_0}$, then there exist directed $P, Q$ with $P, Q <_T P \times Q \equiv_T [\kappa]^{<\mathfrak{n}_0}.$

- (M., 2010) Using these $P$ and $Q$, we can build $X, Y$ such that

$$\chi^{Nt}(X) = \chi^{Nt}(Y) = \mathfrak{n}_1$$

and

$$\chi^{Nt}(X \times Y) = \mathfrak{n}_0.$$

- (Sparado, 2010) $X, Y$ can be modified to get $Z, W$ such that

$$Nt(Z) = Nt(W) = \mathfrak{n}_1$$

and

$$Nt(Z \times W) = \mathfrak{n}_0.$$

- (Spadaro, 2008) There are compact $K, L$ with $Nt(K) = \mathfrak{n}_2,$

$Nt(L) = \mathfrak{n}_3,$ and

$$Nt(K \times L) = \mathfrak{n}_1.$$

- Open: Is $Nt(X^2) \neq Nt(X)$ possible?
Powers

- (M., 2010) We can also use the previous $P$ and $Q$ to build an example of

\[ \chi^N_t (\langle p, q \rangle, X^2) = \aleph_0 < \aleph_1 = \chi^N_t (p, X) = \chi^N_t (q, X). \]
Powers

- (M., 2010) We can also use the previous $P$ and $Q$ to build an example of
  $\chi^N t (\langle p, q \rangle, X^2) = \aleph_0 < \aleph_1 = \chi^N t (p, X) = \chi^N t (q, X)$.
- (M., 2007) If $f : X \rightarrow Y$ is continuous and open at $p$, then
  $\chi^N t (p, X) \leq \chi^N t (f(p), Y)$ (and $\tau(p, X) \geq \tau(f(p), Y)$).

- (M., 2009) If $0 < \gamma < \omega_1$, then
  $\chi^N t (p, X)^\gamma = \chi^N t (p, X)$ and $\chi^N t (X^\gamma) = \chi^N t (X)$.

- However, there are examples of $\chi^N t (X^{\omega_1}) < \chi^N t (X)$ with $\aleph_1 < cf (\chi (X))$.

- (Ridderbos, 2007) If $0 < \gamma < cf (\chi (p, X)))$, then
  $\chi^N t (p^\gamma, X^\gamma) = \chi^N t (p, X)$.

- (M., 2005) If $\chi (p, X) \leq \gamma$ and $|X| > 1$, then
  $\chi^N t (p^\gamma, X^\gamma) = \aleph_0$. 
Powers

▶ (M., 2010) We can also use the previous $P$ and $Q$ to build an example of
\[ \chi^N_t (\langle p, q \rangle, X^2) = \mathbb{N}_0 < \mathbb{N}_1 = \chi^N_t (p, X) = \chi^N_t (q, X). \]

▶ (M., 2007) If $f : X \to Y$ is continuous and open at $p$, then
\[ \chi^N_t (p, X) \leq \chi^N_t (f(p), Y) \text{ (and } \tau(p, X) \geq \tau(f(p), Y)). \]

▶ Hence, $0 < \alpha < \beta \Rightarrow \chi^N_t (p, X^\beta) \leq \chi^N_t (p \upharpoonright \alpha, X^\alpha).$
We can also use the previous $P$ and $Q$ to build an example of
\[ \chi_{Nt}(\langle p, q \rangle, X^2) = \aleph_0 < \aleph_1 = \chi_{Nt}(p, X) = \chi_{Nt}(q, X). \]

If $f : X \to Y$ is continuous and open at $p$, then
\[ \chi_{Nt}(p, X) \leq \chi_{Nt}(f(p), Y) \] (and $\tau(p, X) \geq \tau(f(p), Y)$).

Hence, $0 < \alpha < \beta \Rightarrow \chi_{Nt}(p, X^\beta) \leq \chi_{Nt}(p \upharpoonright \alpha, X^\alpha)$.

If $0 < \gamma < \omega_1$, then $\chi_{Nt}(p^\gamma, X^\gamma) = \chi_{Nt}(p, X)$ and $\chi_{Nt}(X^\gamma) = \chi_{Nt}(X)$.
Powers

- (M., 2010) We can also use the previous P and Q to build an example of
  \[ \chi_{\mathbb{N}_t}(\langle p, q \rangle, X^2) = \aleph_0 < \aleph_1 = \chi_{\mathbb{N}_t}(p, X) = \chi_{\mathbb{N}_t}(q, X). \]

- (M., 2007) If \( f : X \to Y \) is continuous and open at \( p \), then
  \[ \chi_{\mathbb{N}_t}(p, X) \leq \chi_{\mathbb{N}_t}(f(p), Y) \text{ (and } \tau(p, X) \geq \tau(f(p), Y)) \text{.} \]

- Hence, \( 0 < \alpha < \beta \implies \chi_{\mathbb{N}_t}(p, X^\beta) \leq \chi_{\mathbb{N}_t}(p \upharpoonright \alpha, X^\alpha) \).

- (M., 2009) If \( 0 < \gamma < \omega_1 \), then \[ \chi_{\mathbb{N}_t}(p^\gamma, X^\gamma) = \chi_{\mathbb{N}_t}(p, X) \]
  and \[ \chi_{\mathbb{N}_t}(X^\gamma) = \chi_{\mathbb{N}_t}(X). \]

- However, there are examples of \( \chi_{\mathbb{N}_t}(X^{\omega_1}) < \chi_{\mathbb{N}_t}(X) \) with \( \aleph_1 < \text{cf}(\chi(X)) \).

- (Ridderbos, 2007) If \( 0 < \gamma < \text{cf}(\chi(p, X)) \), then
  \[ \chi_{\mathbb{N}_t}(p^\gamma, X^\gamma) = \chi_{\mathbb{N}_t}(p, X). \]

- (M., 2009) If \( \text{cf}(\chi(p, X)) \leq \gamma < \chi(p, X) \), then
  \[ \chi_{\mathbb{N}_t}(p^\gamma, X^\gamma) \leq \chi_{\mathbb{N}_t}(p, X) \leq \chi_{\mathbb{N}_t}(p^\gamma, X^\gamma) +. \]

- (M., 2005) If \( \chi(p, X) \leq \gamma \) and \( |X| > 1 \), then
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Powers

- (M., 2010) We can also use the previous $P$ and $Q$ to build an example of
  \( \chi^{\aleph_0} (\langle p, q \rangle, X^2) = \aleph_0 < \aleph_1 = \chi^{\aleph_0} (p, X) = \chi^{\aleph_0} (q, X) \).

- (M., 2007) If \( f : X \rightarrow Y \) is continuous and open at \( p \), then
  \( \chi^{\aleph_0} (p, X) \leq \chi^{\aleph_0} (f(p), Y) \) (and \( \tau(p, X) \geq \tau(f(p), Y) \)).

- Hence, \( 0 < \alpha < \beta \Rightarrow \chi^{\aleph_0} (p, X^\beta) \leq \chi^{\aleph_0} (p \upharpoonright \alpha, X^\alpha) \).

- (M., 2009) If \( 0 < \gamma < \omega_1 \), then \( \chi^{\aleph_0} (p^\gamma, X^\gamma) = \chi^{\aleph_0} (p, X) \) and \( \chi^{\aleph_0} (X^\gamma) = \chi^{\aleph_0} (X) \).

- However, there are examples of \( \chi^{\aleph_0} (X^{\omega_1}) < \chi^{\aleph_0} (X) \) with \( \aleph_1 < \text{cf} (\chi(X)) \).

- (Ridderbos, 2007) If \( 0 < \gamma < \text{cf} (\chi(p, X)) \), then
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(M., 2010) We can also use the previous $P$ and $Q$ to build an example of
\[ \chi^{\kappa_t}(\langle p, q \rangle, X^2) = \aleph_0 < \aleph_1 = \chi^{\kappa_t}(p, X) = \chi^{\kappa_t}(q, X). \]

(M., 2007) If $f : X \rightarrow Y$ is continuous and open at $p$, then
\[ \chi^{\kappa_t}(p, X) \leq \chi^{\kappa_t}(f(p), Y) \] (and $\tau(p, X) \geq \tau(f(p), Y)$).

Hence, $0 < \alpha < \beta \Rightarrow \chi^{\kappa_t}(p, X^\beta) \leq \chi^{\kappa_t}(p \upharpoonright \alpha, X^\alpha)$.

(M., 2009) If $0 < \gamma < \omega_1$, then $\chi^{\kappa_t}(p^\gamma, X^\gamma) = \chi^{\kappa_t}(p, X)$ and $\chi^{\kappa_t}(X^\gamma) = \chi^{\kappa_t}(X)$.

However, there are examples of $\chi^{\kappa_t}(X^{\omega_1}) < \chi^{\kappa_t}(X)$ with $\aleph_1 < \text{cf}(\chi(X))$.

(Ridderbos, 2007) If $0 < \gamma < \text{cf}(\chi(p, X))$, then
\[ \chi^{\kappa_t}(p^\gamma, X^\gamma) = \chi^{\kappa_t}(p, X). \]

(M., 2009) If $\text{cf}(\chi(p, X)) \leq \gamma < \chi(p, X)$, then
\[ \chi^{\kappa_t}(p^\gamma, X^\gamma) \leq \chi^{\kappa_t}(p, X) \leq \chi^{\kappa_t}(p^\gamma, X^\gamma)^+. \]
(M., 2010) We can also use the previous $P$ and $Q$ to build an example of
\[ \chi_{\mathbb{N}t} (\langle p, q \rangle, X^2) = \aleph_0 < \aleph_1 = \chi_{\mathbb{N}t} (p, X) = \chi_{\mathbb{N}t} (q, X). \]

(M., 2007) If $f : X \to Y$ is continuous and open at $p$, then
\[ \chi_{\mathbb{N}t} (p, X) \leq \chi_{\mathbb{N}t} (f(p), Y) \quad \text{(and } \tau(p, X) \geq \tau(f(p), Y)). \]

Hence, $0 < \alpha < \beta \Rightarrow \chi_{\mathbb{N}t} (p, X^\beta) \leq \chi_{\mathbb{N}t} (p \upharpoonright \alpha, X^\alpha)$. 

(M., 2009) If $0 < \gamma < \omega_1$, then $\chi_{\mathbb{N}t} (p^\gamma, X^\gamma) = \chi_{\mathbb{N}t} (p, X)$ and $\chi_{\mathbb{N}t} (X^\gamma) = \chi_{\mathbb{N}t} (X)$.

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(M., 2009) If $\text{cf}(\chi(p, X)) \leq \gamma < \chi(p, X)$, then
\[ \chi_{\mathbb{N}t} (p^\gamma, X^\gamma) \leq \chi_{\mathbb{N}t} (p, X) \leq \chi_{\mathbb{N}t} (p^\gamma, X^\gamma)^+. \]

(M., 2005) If $\chi(p, X) \leq \gamma$ and $|X| > 1$, then
\[ \chi_{\mathbb{N}t} (p^\gamma, X^\gamma) = \aleph_0. \]
Measuring blossoms

Definition
The \( \lambda \)-wide splitting number at \( p \in X \), or \( \text{split}_\lambda(p, X) \), is the least \( \kappa \) such that \( \tau(p, X) \) has a \((\lambda, \kappa)\)-blossom.
Measuring blossoms

Definition
The $\lambda$-wide splitting number at $p \in X$, or $\text{split}_\lambda(p, X)$, is the least $\kappa$ such that $\tau(p, X)$ has a $(\lambda, \kappa)$-blossom.

Facts (M., 2009)

- $\lambda \leq \mu \Rightarrow \text{split}_\lambda(p, X) \leq \text{split}_\mu(p, X)$. 
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- $\lambda \leq \mu \Rightarrow \text{split}_\lambda(p, X) \leq \text{split}_\mu(p, X)$.
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- $\lambda \leq \mu \Rightarrow \text{split}_\lambda(p, X) \leq \text{split}_\mu(p, X)$.
- $\text{split}_{\chi(p, X)}(p, X) = \chi_{\text{Nt}}(p, X)$.
- $\chi(p, X) < \text{cf} \lambda \Rightarrow \text{split}_\lambda(p, X) = \lambda^+$. 
Measuring blossoms

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The \( \lambda \)-wide splitting number at \( p \in X \), or \( \text{split}_\lambda (p, X) \), is the least \( \kappa \) such that \( \tau (p, X) \) has a \((\lambda, \kappa)\)-blossom.

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- \( \lambda \leq \mu \Rightarrow \text{split}_\lambda (p, X) \leq \text{split}_\mu (p, X) \).
- \( \text{split}_{\chi (p, X)} (p, X) = \chi \text{Nt} (p, X) \).
- \( \chi (p, X) < \text{cf} \lambda \Rightarrow \text{split}_\lambda (p, X) = \lambda^+ \).
- For all singular cardinals \( \lambda \),
  \( \text{split}_\lambda (p, X) \leq (\sup_{\mu < \lambda} \text{split}_\mu (p, X))^+ \).
Measuring blossoms

Definition
The \( \lambda \)-wide splitting number at \( p \in X \), or \( \text{split}_\lambda(p, X) \), is the least \( \kappa \) such that \( \tau(p, X) \) has a \((\lambda, \kappa)\)-blossom.

Facts (M., 2009)

\[ \lambda \leq \mu \implies \text{split}_\lambda(p, X) \leq \text{split}_\mu(p, X). \]
\[ \text{split}_{\chi(p, X)}(p, X) = \chi \text{Nt} (p, X). \]
\[ \chi(p, X) < \text{cf} \lambda \implies \text{split}_\lambda(p, X) = \lambda^+. \]
\[ \text{For all singular cardinals } \lambda, \]
\[ \text{split}_\lambda(p, X) \leq \left( \sup_{\mu < \lambda} \text{split}_\mu(p, X) \right)^+. \]
\[ \text{If } \text{cf} \lambda \leq \kappa < \lambda, \text{ then } \text{split}_\lambda(p^\kappa, X^\kappa) = \sup_{\mu < \lambda} \text{split}_\mu(p, X). \]
What about regular limit cardinals?

**Definition**
Let \( \prod_{i \in I}^{(\kappa)} X_i \) denote the set \( \prod_{i \in I} X_i \) with the topology generated by products of open sets with support smaller than \( \kappa \).
What about regular limit cardinals?

**Definition**
Let $\prod_{i \in I} X_i$ denote the set $\prod_{i \in I} X_i$ with the topology generated by products of open sets with support smaller than $\kappa$.

**Example (M., 2009)**

- If $p \in X = \prod_{\alpha < \lambda}^{(\lambda)} 2^\alpha$ and $\lambda$ is strongly inaccessible, then $\text{split}_\mu(p, X) = \aleph_0$ for all $\mu < \lambda$, but $\text{split}_\lambda(p, X) = \chi \text{Nt}(p, X) = \lambda$.
- The proof’s essential ingredient runs short an elementary-submodel proof of the Erdös-Rado Theorem.
Singular character

Example (M., 2009)

Let \( p \in X = \prod_{\alpha < \omega_1} (\mathbb{N}_1) \prod_{\beta < \omega_1} (\mathbb{N}_\omega) 2. \)
Singular character

Example (M., 2009)

- Let \( p \in X = \prod_{\alpha < \omega_1} (\mathcal{N}_1) \prod_{\beta < \mu} (\mathcal{N}_\omega) \cdot 2. \)
- We then have \( \chi(p, X) = \mathcal{N}_1 \),
  \( \chi_{\text{Nt}} (p, X) = \text{split}_{\omega_1} (p, X) = \mathcal{N}_\omega^+ \), and
  \( \chi_{\text{Nt}} (p^{\omega_1}, X^{\omega_1}) = \sup_{\mu < \omega_1} \text{split}_\mu (p, X) = \mathcal{N}_\omega. \)
Example (M., 2009)

- Let \( p \in X = \prod_{\alpha < \omega_1}^{(\aleph_1)} \prod_{\beta < \aleph_\alpha}^{(\aleph_\omega)} 2 \).
- We then have \( \chi(p, X) = \beth_\omega \),
  \( \chi_{\text{Nt}}(p, X) = \text{split}_{\omega_1}(p, X) = \aleph_\omega^+ \), and
  \( \chi_{\text{Nt}}(p^{\omega_1}, X^{\omega_1}) = \sup_{\mu < \beth_{\omega_1}} \text{split}_\mu(p, X) = \aleph_\omega \).
- The key lemma for the proof is that the set of countably supported maps from \( \omega_1 \) to \( \omega \) (with the product ordering) does not have an \((\omega_1, \aleph_0)\)-blossom.
Singular character

Example (M., 2009)

- Let \( p \in X = \prod_{\alpha < \omega_1} (\aleph_\alpha) \prod_{\beta < \omega} 2. \)
- We then have \( \chi(p, X) = \uparrow_{\omega_1}, \chi_{Nt}(p, X) = \chi_{\text{split}}_{\omega_1}(p, X) = \aleph_\omega, \text{ and } \chi_{Nt}(p^{\omega_1}, X^{\omega_1}) = \sup_{\mu < \omega_1} \chi_{\text{split}}_{\mu}(p, X) = \aleph_\omega. \)
- The key lemma for the proof is that the set of countably supported maps from \( \omega_1 \) to \( \omega \) (with the product ordering) does not have an \((\omega_1, \aleph_0)\)-blossom.
- Why? If \( F : \omega_1 \to F_n(\omega_1, \omega, \aleph_1), F \in M \prec H(\aleph_2), \) and \( |M| = \aleph_0, \) then we can use reflect properties of \( F(\omega_1 \cap M) \) to find infinitely many \( F(\alpha) \in M \) all dominated by a single \( g \in F_n(\omega_1, \omega, \aleph_1). \)
Example (M., Spadaro, 2009)

Let $p \in X = \prod_{\alpha < \aleph_\omega}^{(\aleph_1)} 2$. We then have
\[ \chi(p, X) = \pi(X) = w(X) = \aleph_\omega^{\aleph_0}. \]
Example (M., Spadaro, 2009)

Let \( p \in X = \prod_{\alpha < \aleph_\omega}^{(\aleph_1)} 2 \). We then have
\[
\chi(p, X) = \pi(X) = w(X) = \aleph_{\omega}^{\aleph_0}.
\]

\( \aleph_1 \leq Nt(X) \leq c^+ \). Moreover, \( c \leq \aleph_{\omega+1} \Rightarrow Nt(X) \leq \aleph_{\omega+1} \).

Open: can we have \( Nt(X) > \aleph_{\omega+1} \)?
Example (M., Spadaro, 2009)

Let \( p \in X = \prod_{\alpha < \aleph_\omega} \mathbb{N}_1 \). We then have
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\chi(p, X) = \pi(X) = w(X) = \aleph_{\omega_0}.
\]

\( \aleph_1 \leq Nt(X) \leq c^+ \). Moreover, \( c \leq \aleph_{\omega+1} \Rightarrow Nt(X) \leq \aleph_{\omega+1} \).
Open: can we have \( Nt(X) > \aleph_{\omega+1} \)?

If \( \square_{\aleph_\omega} \) and \( \aleph_{\omega_0} = \aleph_{\omega+1} \), then
\[
Nt(X) = \pi Nt(X) = \chi Nt(p, X) = \aleph_1.
\]
(Why? We can use Bernstein sets and a locally countable \( S \subseteq [\aleph_\omega]^{\aleph_0} \) of size \( \aleph_{\omega+1} \) to build an \( \aleph_1 \)-short base...
Example (M., Spadaro, 2009)

- Let \( p \in X = \prod_{\alpha < \aleph_\omega}^{(\aleph_1)} 2 \). We then have 
  \[ \chi(p, X) = \pi(X) = w(X) = \aleph_\omega^{\aleph_0}. \]

- \( \aleph_1 \leq \text{Nt}(X) \leq c^+ \). Moreover, \( c \leq \aleph_{\omega+1} \Rightarrow \text{Nt}(X) \leq \aleph_{\omega+1}. \) Open: can we have \( \text{Nt}(X) > \aleph_{\omega+1} \)?

- If \( \Box_{\aleph_\omega} \) and \( \aleph_\omega^{\aleph_0} = \aleph_{\omega+1} \), then 
  \[ \text{Nt}(X) = \pi \text{Nt}(X) = \chi \text{Nt}(p, X) = \aleph_1. \) (Why? We can use Bernstein sets and a locally countable \( S \subseteq [\aleph_\omega]^{\aleph_0} \) of size \( \aleph_{\omega+1} \) to build an \( \aleph_1 \)-short base...)

- (Soukup)
  \( (\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0) \Rightarrow \text{Nt}(X) \geq \chi \text{Nt}(p, X) \geq \aleph_2. \) (The hypothesis is consistent relative (roughly) to a huge cardinal (Levinski, Magidor, Shelah, 1990).)
Example (M., Spadaro, 2009)

- Let \( p \in X = \prod_{\alpha < \aleph_\omega}^{(\aleph_1)} 2 \). We then have
  \[ \chi(p, X) = \pi(X) = w(X) = \aleph_\omega^{\aleph_0}. \]

- \( \aleph_1 \leq Nt(X) \leq c^+ \). Moreover, \( c \leq \aleph_{\omega+1} \Rightarrow Nt(X) \leq \aleph_{\omega+1} \).
  Open: can we have \( Nt(X) > \aleph_{\omega+1} \)?

- If \( \square_{\aleph_\omega} \) and \( \aleph_\omega^{\aleph_0} = \aleph_{\omega+1} \), then
  \[ Nt(X) = \pi Nt(X) = \chi Nt(p, X) = \aleph_1. \] (Why? We can use Bernstein sets and a locally countable \( S \subseteq [\aleph_\omega]^{\aleph_0} \) of size \( \aleph_{\omega+1} \) to build an \( \aleph_1 \)-short base . . .)

- (Soukup)
  \[ (\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0) \Rightarrow Nt(X) \geq \chi Nt(p, X) \geq \aleph_2. \] (The hypothesis is consistent relative (roughly) to a huge cardinal (Levinski, Magidor, Shelah, 1990).)

- Open: Can we have \( \pi Nt(X) > \aleph_1 \)? Equivalently, can \( \langle F_n(\aleph_\omega, 2, \aleph_1), \subseteq \rangle \) fail to be almost \( \aleph_1 \)-short?
Noetherian spectra

Another application of Bernstein sets (M., 2009)
If \( c \geq \kappa \) and \( \kappa \) is weakly inaccessible, then there is a Lindelöf linear order with Noetherian type \( \kappa \).

Excluded Noetherian types (M., 2008)

- The compact linear orders attain all Noetherian types except \( \aleph_1 \) and weak inaccessibles.
Noetherian spectra

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If $c \geq \kappa$ and $\kappa$ is weakly inaccessible, then there is a Lindelöf linear order with Noetherian type $\kappa$.

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- The compact linear orders attain all Noetherian types except $\aleph_1$ and weak inaccessibles.
- The dyadic compacta do not attain Noetherian type $\aleph_1$. 
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Excluded Noetherian types (M., 2008)

- The compact linear orders attain all Noetherian types except $\aleph_1$ and weak inaccessibles.
- The dyadic compacta do not attain Noetherian type $\aleph_1$.
- Open: do the dyadic compacta attain weakly inaccessible Noetherian types?
- Open: do the dyadic compacta attain Noetherian type $\aleph_{\omega+1}$?
Local bases in $\beta \omega \setminus \omega$

Convention

- If $\mathcal{U}$ is an ultrafilter on $\omega$, then order $\mathcal{U}$ by $\supseteq$.
- Let $\mathcal{U}_*$ denote $\mathcal{U}$ ordered by $\supseteq^*$ (containment modulo $[\omega]^{<\aleph_0}$).

Isbell's Problem

ZFC proves there exists $\mathcal{U} \in \beta \omega \setminus \omega$ such that $\mathcal{U}^* \equiv T \equiv T[c] < \aleph_0$.

Does ZFC prove there exists $\mathcal{V} \in \beta \omega \setminus \omega$ such that $\mathcal{V} \not\equiv T[c] < \aleph_0$?
Local bases in $\beta\omega \setminus \omega$

Convention

- If $U$ is an ultrafilter on $\omega$, then order $U$ by $\supseteq$.
- Let $U_*$ denote $U$ ordered by $\supseteq^*$ (containment modulo $[\omega]^{<\aleph_0}$).

Facts

- Given $U \in \beta\omega \setminus \omega$, $\tau(U, \beta\omega \setminus \omega)$ is mutually cofinal with $U_*$.
- Hence, $U$ has a flat local base in $\beta\omega \setminus \omega$ if and only if $U_* \geq_T [\chi(U, \beta\omega \setminus \omega)]^{<\aleph_0}$.
Local bases in $\beta\omega \setminus \omega$

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- If $\mathcal{U}$ is an ultrafilter on $\omega$, then order $\mathcal{U}$ by $\supseteq$.
- Let $\mathcal{U}_*$ denote $\mathcal{U}$ ordered by $\supseteq^*$ (containment modulo $[\omega]^{<\aleph_0}$).

Facts

- Given $\mathcal{U} \in \beta\omega \setminus \omega$, $\tau(\mathcal{U}, \beta\omega \setminus \omega)$ is mutually cofinal with $\mathcal{U}_*$.
- Hence, $\mathcal{U}$ has a flat local base in $\beta\omega \setminus \omega$ if and only if $\mathcal{U}_* \geq_T [\chi(\mathcal{U}, \beta\omega \setminus \omega)]^{<\aleph_0}$.
- Likewise, $\mathcal{U}$ has a flat local base in $\beta\omega$ if and only if $\mathcal{U} \geq_T [\chi(\mathcal{U}, \beta\omega \setminus \omega)]^{<\aleph_0}$. 

Isbell's Problem

ZFC proves there exists $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\mathcal{U} \not\equiv_T c < \aleph_0$.

Does ZFC prove there exists $\mathcal{V} \in \beta\omega \setminus \omega$ such that $\mathcal{V} \not\equiv_T c < \aleph_0$?
Local bases in $\beta\omega \setminus \omega$

Convention

- If $\mathcal{U}$ is an ultrafilter on $\omega$, then order $\mathcal{U}$ by $\supseteq$.
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Facts

- Given $\mathcal{U} \in \beta\omega \setminus \omega$, $\tau(\mathcal{U}, \beta\omega \setminus \omega)$ is mutually cofinal with $\mathcal{U}_*$.
- Hence, $\mathcal{U}$ has a flat local base in $\beta\omega \setminus \omega$ if and only if $\mathcal{U}_* \geq_T [\chi(\mathcal{U}, \beta\omega \setminus \omega)]^{<\aleph_0}$.
- Likewise, $\mathcal{U}$ has a flat local base in $\beta\omega$ if and only if $\mathcal{U} \geq_T [\chi(\mathcal{U}, \beta\omega \setminus \omega)]^{<\aleph_0}$.

Isbell’s Problem

ZFC proves there exists $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\mathcal{U}_* \equiv_T \mathcal{U} \equiv_T [c]^{<\aleph_0}$. Does ZFC prove there exists $\mathcal{V} \in \beta\omega \setminus \omega$ such that $\mathcal{V} \not\equiv_T [c]^{<\aleph_0}$?
U versus $U_*$

- This seminar has already heard a lot about recent progress for Tukey classes of the form $U$ by Dobrinen, Raghavan, and Todorčević. I will focus on Tukey classes of the form $U_*$.
This seminar has already heard a lot about recent progress for Tukey classes of the form $\mathcal{U}$ by Dobrinen, Raghavan, and Todorčević. I will focus on Tukey classes of the form $\mathcal{U}_*$. 

$\mathcal{U}_* \leq_T \mathcal{U} \leq_T [\mathcal{c}]^{<\aleph_0}$ for all $\mathcal{U} \in \beta\omega \setminus \omega$. 

Hence, Isbell’s Problem is equivalent to asking if ZFC proves there exists $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\mathcal{U}_* \not\equiv_T [\mathcal{c}]^{<\aleph_0}$. 


This seminar has already heard a lot about recent progress for Tukey classes of the form $\mathcal{U}$ by Dobrinen, Raghavan, and Todorčević. I will focus on Tukey classes of the form $\mathcal{U}_*$. 

- $\mathcal{U}_* \leq_T \mathcal{U} \leq_T [\mathfrak{c}]^{<\aleph_0}$ for all $\mathcal{U} \in \beta\omega \setminus \omega$.
- If $\mathcal{U}_*$ is not $\aleph_1$-directed, then $\mathcal{V} \leq_T \mathcal{U}_*$ for some $\mathcal{V} \in \beta\omega \setminus \omega$. 
This seminar has already heard a lot about recent progress for Tukey classes of the form \( \mathcal{U} \) by Dobrinen, Raghavan, and Todorčević. I will focus on Tukey classes of the form \( \mathcal{U}_* \).

\( \mathcal{U}_* \leq_T \mathcal{U} \leq_T [\mathfrak{c}]^{<\aleph_0} \) for all \( \mathcal{U} \in \beta\omega \setminus \omega \).

If \( \mathcal{U}_* \) is not \( \aleph_1 \)-directed, then \( \mathcal{V} \leq_T \mathcal{U}_* \) for some \( \mathcal{V} \in \beta\omega \setminus \omega \).

If \( P \) is \( \aleph_1 \)-directed and \( \kappa \geq \aleph_0 \), then \( P \not\geq_T [\kappa]^{<\aleph_0} \).

Hence, Isbell's Problem is equivalent to asking if ZFC proves there exists \( \mathcal{U} \in \beta\omega \setminus \omega \) such that \( \mathcal{U}_* \not\equiv_T [\mathfrak{c}]^{<\aleph_0} \).
This seminar has already heard a lot about recent progress for Tukey classes of the form $U$ by Dobrinen, Raghavan, and Todorčević. I will focus on Tukey classes of the form $U_*$. 

- $U_* \leq_T U \leq_T [c]^{<\aleph_0}$ for all $U \in \beta\omega \setminus \omega$.
- If $U_*$ is not $\aleph_1$-directed, then $V \leq_T U_*$ for some $V \in \beta\omega \setminus \omega$.
- If $P$ is $\aleph_1$-directed and $\kappa \geq \aleph_0$, then $P \not\geq_T [\kappa]^{<\aleph_0}$.
- Hence, Isbell’s Problem is equivalent to asking if ZFC proves there exists $U \in \beta\omega \setminus \omega$ such that $U_* \not\equiv_T [c]^{<\aleph_0}$. 

Ultrafilter Tukey classes for $\supseteq^*$

- (M., 2008) Assuming $p = c$, for every regular $\kappa \in [\aleph_0, c]$, there exists $U_* \equiv_T [c]^{<\kappa}$, which implies $\chi_{\text{Nt}} (U, \beta\omega \setminus \omega) = \kappa$. 

- (Aviles, Todorčević, 2010) If $n \geq 2$, $\kappa < m$ $\sigma$-linked, and $A_0, \ldots, A_n \subseteq \beta\omega \setminus \omega$ are disjoint open $\kappa$ sets, then there are clopen $B_0 \supseteq A_0, \ldots, B_n \supseteq A_n$ such that $\bigcap_{i\leq n} B_i = \emptyset$.

- (M., 2009) Assuming $t = c$ and $\Diamond (S_0, c^1)$ (which are implied by $\text{MA} \land c = \aleph_2$), there exists $W : 2^c \to \beta\omega \setminus \omega$ such that $W(f) \upharpoonright \supseteq^* W(g)$ for all $f \neq g$.

- Open: Does CH imply there exist $U, V \in \beta\omega \setminus \omega$ such that $U \supseteq^* V \supseteq^* U$?
Ultrafilter Tukey classes for $\supseteq^*$

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- (Aviles, Todorčević, 2010) If $n \geq 2$, $\kappa < m_{\sigma-n\text{-linked}}$, and $A_0, \ldots, A_n \subseteq \beta \omega \setminus \omega$ are disjoint open $F_\kappa$ sets, then there are clopen $B_0 \supseteq A_0, \ldots, B_n \supseteq A_n$ such that $\bigcap_{i \leq n} B_i = \emptyset$.

- (M., 2010) It follows that $U_* \not\equiv_T \kappa \times P$ for all $U \in \beta \omega \setminus \omega$ if $\omega \leq \text{cf}(\kappa) = \kappa < \sup_{n < \omega} m_{\sigma-n\text{-linked}}$ and $P$ is the union of at most $\kappa$-many $\kappa^{+}$-directed sets. E.g., $U_* \not\equiv_T \omega \times \omega_1$, and $\text{MA}_{\aleph_1} \Rightarrow U_* \not\equiv_T \omega_1 \times \omega_2$. 
Ultrafilter Tukey classes for $\supseteq^*$

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- Open: Does CH imply there exist $U, V \in \beta \omega \setminus \omega$ such that $U_* \not\leq_T V^*$? 

Ultrafilter Tukey classes for $\supseteq^*$

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- Open: Does CH imply there exist $U, V \in \beta \omega \setminus \omega$ such that $U_* \not\leq_T V_* \not\leq_T U_*$?
The (local) Noetherian ($\pi$-)type of $\beta \omega \setminus \omega$

ZFC proves each of the following statements.

- $\pi \text{Nt} (\beta \omega \setminus \omega) = h \leq s \leq \text{Nt} (\beta \omega \setminus \omega) \leq c^+$.
- $\chi \text{Nt} (\beta \omega \setminus \omega) \leq \min \{\text{Nt} (\beta \omega \setminus \omega), c\}$.
- $\text{MA} \Rightarrow \pi \text{Nt} (\beta \omega \setminus \omega) = c \Rightarrow \text{Nt} (\beta \omega \setminus \omega) = c$.
- $r = c \Rightarrow \text{Nt} (\beta \omega \setminus \omega) \leq c$.
- $r < c \Rightarrow \text{Nt} (\beta \omega \setminus \omega) \geq c$.
- $r < c \text{f} c \Rightarrow \text{Nt} (\beta \omega \setminus \omega) = c^+$.

Each of the following statements is consistent with ZFC.

- $\omega_1 = \pi \text{Nt} (\beta \omega \setminus \omega) = \chi \text{Nt} (\beta \omega \setminus \omega) = \text{Nt} (\beta \omega \setminus \omega) < c$.
- $\omega_1 < \pi \text{Nt} (\beta \omega \setminus \omega) = \chi \text{Nt} (\beta \omega \setminus \omega) = \text{Nt} (\beta \omega \setminus \omega) < c$.
- $\omega_1 = \pi \text{Nt} (\beta \omega \setminus \omega) < \text{Nt} (\beta \omega \setminus \omega) < c$.
- $\omega_1 < \pi \text{Nt} (\beta \omega \setminus \omega) < \chi \text{Nt} (\beta \omega \setminus \omega) = c < \text{Nt} (\beta \omega \setminus \omega)$.