Davies trees and stratified inverse limits

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Some pre-Cohen set theory

Theorem

CH implies that ${}^2\mathbb{R}$ is a countable union of rotations of graphs of functions.

Proof.

- Replace ${}^2\mathbb{R}$ with ${}^2\omega_1$.
- Let $\varphi_\alpha \colon \omega \to \alpha + 1$ be surjective for all $\alpha < \omega_1$.
- Let $f_n(\alpha) = \varphi_\alpha(n)$ for all $n < \omega$ and $\alpha < \omega_1$.
- ${}^2\omega_1 = \bigcup_{n<\omega}(f_n \cup f_n^{-1})$.

Question (Sierpinski, 1951). Is the converse true?

Theorem (Davies, 1963)

ZFC already implies that ${}^2\mathbb{R}$ is a countable union of rotations of graphs of functions.

Davies' proof

Actually, Davies proved something stronger:

Theorem

Let F be an infinite field and let $(L_n : n < \omega)$ be a sequence of pairwise non-parallel lines in ${}^2F$. Then ${}^2F$ can be partitioned into sets $(S_n : n < \omega)$ such that for all $n < \omega$ and all lines $L \parallel L_n$, $|S_n \cap L| \leq 1$.

Proof.

1. Let $\mathfrak{A} = (H(\theta), \in, F, +, \cdot, \vec{L})$ and find $(M_\alpha : \alpha < \kappa)$ such that
    1.1 ${}^2F \subseteq \bigcup_{\alpha<\kappa} M_\alpha$,
    1.2 each M_α is a countable and $M_\alpha \prec \mathfrak{A}$, and
    1.3 each $\bigcup_{\beta<\alpha} M_\beta$ is a **finite** union $\bigcup_{i<m_\alpha} N_\alpha^i$ where $N_\alpha^i \prec \mathfrak{A}$. where each $N_\alpha^i \prec \mathfrak{A}$.
2. Ignore M_α if ${}^2F \cap M_\alpha \subseteq \bigcup_{i<m_\alpha} N_\alpha^i$.
3. Otherwise, let ${}^2F \cap M_\alpha \setminus \bigcup_{i<m_\alpha} N_\alpha^i = \{p_k : k < \omega\}$.
4. For each p_k and N_α^i, there is at most one $n(k,i)$ such that the line through p_k parallel to $L_{n(k,i)}$ intersects N_α^i.
5. Put p_k in $S_{r(k)}$ where $r(k) \neq n(k,i)$ for all $i < m_\alpha$ and $r(k) \neq r(j)$ for all $j < k$.

The tree

To achieve the finite union of 1.3, Davies constructs $\vec{M}$ as the leaves of what I will call a **Davies tree**.

- A Davies tree is of the form $(\mathfrak{A}_t : t \in T)$ where T is a well-founded tree of finite sequences of ordinals.
- The lexicographic ordering of the leaves of any such T is necessarily a well-ordering.
- ${}^2F \subseteq \mathfrak{A}_\varnothing \prec \mathfrak{A}$.
- If $\mathfrak{A}_t$ is countable, then t is a leaf of T.
- If $\mathfrak{A}_t$ is uncountable, then
  - $\mathfrak{A}_t = \bigcup\{\mathfrak{A}_{t^\frown\alpha} : t^\frown\alpha \in T\}$,
  - $\alpha < \beta \Rightarrow \mathfrak{A}_{t^\frown\alpha} \subseteq \mathfrak{A}_{t^\frown\beta}$,
  - $\mathfrak{A}_{t^\frown\alpha} \prec \mathfrak{A}_t$, and
  - $|\mathfrak{A}_{t^\frown\alpha}| < |\mathfrak{A}_t|$.
- T is well-founded because $s \subset t$ implies $|\mathfrak{A}_s| > |\mathfrak{A}_t|$.

The finite union

- Let $t = (\alpha_0, \dots, \alpha_{m_\alpha - 1})$ be a leaf of T.
- Every leaf $s <_{\mathrm{lex}} t$ is of the form $(\alpha_0, \dots, \alpha_{i-1}, \beta, \gamma_{i+1}, \dots, \gamma_{n-1})$ where $i < m_\alpha$ and $\beta < \alpha_i$.
- Set $p(t, i, \beta) = (\alpha_0, \dots, \alpha_{i-1}, \beta)$.
- Set $\mathfrak{B}_{t,i} = \bigcup_{\beta < \alpha_i} \mathfrak{A}_{p(t,i,\beta)}$.
- $\mathfrak{B}_{t,i} \prec \mathfrak{A}$ or $\mathfrak{B}_{t,i} = \emptyset$.
- $\bigcup_{s <_{\mathrm{lex}} t} \mathfrak{A}_s = \bigcup_{i < m_\alpha} \mathfrak{B}_{t,i}$.

Costs and benefits

Benefits

- We can construct something arbitrarily large one countable piece at a time: $(S_n \cap M_\alpha : n < \omega)$ for $\alpha < \kappa$.
- Our work done prior to handling M_α is a finite union of very nice pieces: $(S_n \cap N^i_\alpha : n < \omega)$ for $i < m_\alpha$.

Unavoidable Costs

- The nice pieces might interact in nasty ways because generally $N^i_\alpha \not\subseteq N^j_\alpha$.
- Davies trees only work in contexts where these interactions are sufficiently benign.

Avoidable costs

- "$(N^i_\alpha : i < m_\alpha) \in M_\alpha$" is easy to arrange, but "$(S_n \cap N^i_\alpha : i < m_\alpha \wedge n < \omega) \in M_\alpha$" is not.
- This could be a problem in some contexts.
- The work-around is to build $\vec{S}$ and the Davies tree simultaneously...

Long ω_1-approximation sequences

(M., 2008) There is a simpler structure that induces a Davies tree.

- Let $\mathcal{L}$ be a countable language extending $\{\in\}$.
- Let $\mathfrak{A}$ be an $\mathcal{L}$-expansion of $(H(\theta), \in)$.
- Let $(M_\alpha : \alpha < \eta)$ be a **long ω_1-approximation sequence**:
  - M_α is countable and $M_\alpha \prec \mathfrak{A}$.
  - $(M_\beta : \beta < \alpha) \in M_\alpha$.
- It is easy to build $\vec{M}$ and $\vec{S}$ simultaneously.
- Warning: If $\alpha \geq \omega_1$, then $\alpha \not\subseteq M_\alpha$ and $\exists \beta < \alpha \;\; M_\beta \notin M_\alpha \;\wedge\; M_\beta \not\subseteq M_\alpha$.
- There is $\varnothing$-definable well-founded class tree Υ of finite sequences of ordinals such that if η is a cardinal and the first η leaves of Υ, according to the lexicographic ordering, are $(u_\alpha : \alpha < \eta)$, then $(\mathfrak{A}_t : t \in T)$ is a Davies tree where:
  - $T = \{t : \exists \alpha < \eta \;\; t \subseteq u_\alpha\}$.
  - $\mathfrak{A}_{u_\alpha} = M_\alpha$.
  - $\mathfrak{A}_t = \bigcup\{\mathfrak{A}_{t^\frown\alpha} : t^\frown\alpha \in T\}$ if t is not a leaf.

Ordinal division

- $\forall \alpha, \beta > 0 \;\; \exists! \gamma, \delta \quad \alpha = \beta \cdot \gamma + \delta \;\wedge\; \delta < \beta$.
- $\forall \alpha > 0 \;\; \exists! Q\alpha, R\alpha \quad \alpha = |\alpha| \cdot Q\alpha + R\alpha \;\wedge\; R\alpha < |\alpha|$.

Repeatedly divide the remainder by its cardinality and call the result the **cardinal normal form** of α:

$$\alpha = |\alpha| \cdot Q\alpha \quad + R\alpha$$

$$\alpha = |\alpha| \cdot Q\alpha \quad + |R\alpha| \cdot QR\alpha \quad + R^2\alpha$$

$$\vdots$$

$$\alpha = |\alpha| \cdot Q\alpha \quad + |R\alpha| \cdot QR\alpha \quad + \left|R^2\alpha\right| \cdot QR^2\alpha + \cdots + R^{m_\alpha}\alpha$$

Stop when $R^{m_\alpha}\alpha$ is countable.
Edge cases: $R^0 = \mathrm{id}$; $m_\alpha = 0$ for all $\alpha < \omega_1$.

A canonical Davies tree

- The non-leaves of Υ are the sequences $(\tau_\alpha^j : j < i)$, with $i \leq m_\alpha$, of **truncations** $\tau_\alpha^j = \sum_{k<j} \left|R^k\alpha\right| \cdot QR^k\alpha$.
- For convenience, set $\tau_\alpha^{m_\alpha} = \alpha$.
- The leaves are the sequences $(\tau_\alpha^j : j \leq m_\alpha)$.

With the above machinery, the Davies tree remains a very useful intuition pump, but becomes formally unnecessary:
Given a long ω_1-approximation sequence $(M_\alpha : \alpha < \eta)$,

- the **eras** of α are the intervals $I_\alpha^i = [\tau_\alpha^i, \tau_\alpha^{i+1})$ where $i < m_\alpha$;
- the **strata** of M_α are the unions $N_\alpha^i = \bigcup\{M_\beta : \beta \in I_\alpha^i\}$;
- $\bigcup_{\beta<\alpha} M_\beta = \bigcup_{i<m_\alpha} N_\alpha^i$;
- $N_\alpha^i \in M_\alpha$ and $\left|N_\alpha^i\right| \subseteq N_\alpha^i \prec \mathfrak{A}$;
- $i < j < m_\alpha \;\Rightarrow\; N_\alpha^i \in N_\alpha^j \;\wedge\; \left|N_\alpha^i\right| > \left|N_\alpha^j\right|$;
- N_α^i is uncountable, except possibly when $i = m_\alpha - 1$.

Another application of Davies trees

Theorem (Jackson and Mauldin, 2002)

There exists $S \subseteq {}^2\mathbb{R}$ *such that* $|S \cap L| = 1$ *for every lattice* L *isometric with* ${}^2\mathbb{Z}$*.*

- Jackson and Mauldin explicitly use a Davies tree in order to proceed one countable piece at a time, organizing all prior work into finitely many nice pieces.
- The Jackson-Mauldin result is much harder than Davies' result because the class of finite configurations of points that must be avoided is much more complicated.

An implicit application to group theory

Theorem (Shelah, 1975)

Let G be a group and λ a singular cardinal such that every subgroup of G of size less than λ is free. Every subgroup of G of size λ is then also free.

- The above theorem is just a special case of Shelah's compactness theorem for singular cardinals.
- Shelah doesn't explicitly use a Davies tree, but he implicitly uses the first three non-root levels of a Davies tree, avoiding higher levels through an intricate inductive argument.

Openly generated compacta

- Let X be a compactum (*i.e.*, a compact Hausdorff space).
- Let $C(X)$ be the algebra of all continuous $f\colon X \to \mathbb{R}$.
- Given a class A, let X/A denote the quotient space where points are identified iff no $f \in C(X) \cap A$ distinguishes them.
- **Quotient characterization (essentially Ščepin, 1981):** X is openly generated iff, for all $M \prec (H(\theta), \in, C(X))$, the quotient map $q^X_M\colon X \to X/M$ is open.
- **Example (easy):** Powers of 2 are openly generated.
- **Example (Šapiro, 1976):** The Vietoris hyperspace of ${}^{\aleph_2}2$ is openly generated but not a continuous image of a power of 2.
- **Theorem (Ščepin, 1979).** If Y is a continuous image of an openly generated compactum, then Y is ccc and $\pi\chi(Y) = w(Y)$.

Flat bases

- A (local) base of a space is **flat** if every element of the (local) base has only finitely many supersets in the (local) base.
- (M., 2008) If Y is a continuous image of an openly generated compactum X, and $\mathcal{A}$ is a local base in Y, then $\mathcal{A}$ contains a flat local base $\mathcal{B}$ in Y.
  - This result didn't need a long ω_1-approximation sequence for its proof, just a continuous elementary $\in$-chain.
- A space Y is **homogeneous** if for all $p, q \in Y$, there is a homeomorphism $f \colon Y \to Y$ such that $f(p) = q$.
- (M., 2008) If Y is a homogeneous continuous image of an openly generated compactum X, and $\mathcal{A}$ is a base of Y, then $\mathcal{A}$ contains a flat base $\mathcal{B}$ of Y.
  - The result was proved with a long ω_1-approximation sequence.
  - A stronger result for metrizable compacta was used to ensure that the inductive construction succeeded.
  - The crucial inductive argument is that if each $\mathcal{B} \cap N^i_\alpha$ is flat, then, for all nonempty open U, $q^Y_{N^i_\alpha}[U]$ has only finitely many supsersets in $\mathcal{B} \cap \bigcup_{\beta<\alpha} M_\alpha$.

Why flatness?

(M., 2008) Every known homogeneous compactum (including all compact groups) has a flat local base.
(By homogeneity, this is equivalent to saying that every local base contains a flat local base.)

Conjecture. Every homogeneous compactum has a flat local base.

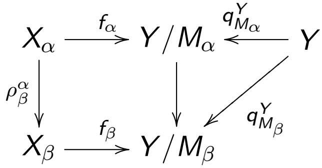
Why homogeneity?

- The **cellularity** $c(X)$ of a space X is the supremum of the cardinalities of its pairwise disjoint families of open sets.
- **Van Douwen's Problem (c. 1970).** Is there a homogeneous compactum with cellularity greater than $\mathfrak{c}$?
- Van Douwen's Problem is still open in all models of ZFC.
- A (local) base is κ-flat if every element of the (local) base has fewer than κ-many supersets in the (local) base.
- (M., 2008) If GCH holds, then, for all homogeneous compacta X, every local base in X contains a $c(X)$-flat local base.
- This is weak evidence for "no" to Van Douwen. Nevertheless...
- **Conjecture.** There are homogeneous compacta X with $c(X) > \mathfrak{c}$ because every compactum Y is a continuous image of a homogeneous compactum X.

A new application to homogeneous compacta

(M., 2012) If W is a zero-dimensional openly generated compactum, then W is a continuous open image of a homogeneous zero-dimensional openly generated compactum Z.

- The continuous surjection $g\colon Z \to W$ is found by building a certain inverse limit $Z = \varprojlim_{\xi<\lambda} Z_\xi$ with $Z_0 = 1$ and $Z_1 = W$.
- $Z_{\xi+1}$ is a fiber product of the form $Z_\xi \times_{Z_\zeta} K_\xi$ where $\zeta < \xi$. K_ξ and the map $f\colon K_\xi \to Z_\zeta$ are built using a long ω_1-approximation sequence $(M_\alpha : \alpha < w(Z_\zeta))$.
- Letting $X = K_\xi$ and $Y = Z_\zeta$, we build X and f as inverse limits. With X_α always 2^ω, we build $f_\alpha\colon X_\alpha \to Y/M_\alpha$ and bonding maps $\rho^\alpha_\beta\colon X_\alpha \to X_\beta$ whenever $M_\beta \subsetneq M_\alpha$, such that:



Stratified inverse limits

The pattern of construction of X can be axiomatized to yield a new kind of inverse limit construction.

- If κ is a cardinal and $(M_\alpha : \alpha < \kappa)$ is a long ω_1-approximation sequence, then $\in \restriction \{M_\alpha : \alpha < \kappa\}$ is transitive and *directed*.
- An object X (in some complete category) will be built as the inverse limit of a sequence $(X_\alpha : \alpha < \kappa)$ with morphisms $\rho^\alpha_\beta \colon X_\alpha \to X_\beta$ for all pairs $M_\beta \in M_\alpha$, such that $\rho^\beta_\gamma \circ \rho^\alpha_\beta = \rho^\alpha_\gamma$ for all triples $M_\gamma \in M_\beta \in M_\alpha$.
- $M_\beta \in M_\alpha \Leftrightarrow M_\beta \subsetneq M_\alpha \Leftrightarrow \beta \in \alpha \cap M_\alpha$.
- Proceeding by induction on α, suppose we have a partial inverse system $((X_\beta : \beta < \alpha), (\rho^\beta_\gamma : M_\gamma \in M_\beta \wedge \beta < \alpha))$.
- $\in \restriction \{M_\beta : \beta < \alpha\}$ is not directed in general, but $\in \restriction \{M_\beta : \beta \in \alpha \cap N^i_\alpha\}$ is directed for each stratum N^i_α.
- By elementarity, $\in \restriction \{M_\beta : \beta \in \alpha \cap N^i_\alpha \cap M_\alpha\}$ is also directed.

Extending the partial inverse system

Choose an object X_α and *finitely many* morphisms σ_i^α such that the following diagram commutes for all $i, j < m_\alpha$.

$$\begin{array}{ccc}
X_{\alpha,i} = \varprojlim_{\beta \in \alpha \cap N_\alpha^i \cap M_\alpha} X_\beta & \xleftarrow{\sigma_i^\alpha} & X_\alpha \\
\downarrow & & \downarrow \sigma_j^\alpha \\
\varprojlim_{\beta \in \alpha \cap N_\alpha^i \cap N_\alpha^j \cap M_\alpha} X_\beta & \longleftarrow & X_{\alpha,j} = \varprojlim_{\beta \in \alpha \cap N_\alpha^j \cap M_\alpha} X_\beta
\end{array}$$

This is equivalent to choosing an object X_α and a *single* morphism from X_α to the fiber product of $X_{\alpha,0}, \dots, X_{\alpha,m_\alpha - 1}$.
For each $M_\beta \in M_\alpha$, choose i such that $\beta \in N_\alpha^i$ and define ρ_β^α as the natural composite morphism.

$$\begin{array}{ccc}
X_{\alpha,i} & \xleftarrow{\sigma_i^\alpha} & X_\alpha \\
\downarrow & \swarrow \rho_\beta^\alpha & \\
X_\beta & &
\end{array}$$