

Davies trees and their applications

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Some pre-Cohen set theory

Theorem

CH implies that ${}^2\mathbb{R}$ is a countable union of rotations of graphs of functions.

Proof.

- ▶ Replace ${}^2\mathbb{R}$ with ${}^2\omega_1$.
- ▶ Let $\varphi_\alpha: \omega \rightarrow \alpha + 1$ be surjective for all $\alpha < \omega_1$.
- ▶ Let $f_n(\alpha) = \varphi_\alpha(n)$ for all $n < \omega$ and $\alpha < \omega_1$.
- ▶ ${}^2\omega_1 = \bigcup_{n < \omega} (f_n \cup f_n^{-1})$.

Question (Sierpinski, 1951). Is the converse true?

Theorem (Davies, 1963)

ZFC already implies that ${}^2\mathbb{R}$ is a countable union of rotations of graphs of functions.

Davies' proof

Actually, Davies proved something stronger:

Theorem

Let F be an infinite field and let $(L_n : n < \omega)$ be a sequence of pairwise non-parallel lines in 2F . Then 2F can be partitioned into sets $(S_n : n < \omega)$ such that for all $n < \omega$ and all lines $L \parallel L_n$, $|S_n \cap L| \leq 1$.

Proof.

1. Let $\mathfrak{A} = (H(\theta), \in, F, +, \cdot, \vec{L})$ and find $(M_\alpha : \alpha < \kappa)$ such that
 - 1.1 ${}^2F \subseteq \bigcup_{\alpha < \kappa} M_\alpha$,
 - 1.2 each M_α is a countable and $M_\alpha \prec \mathfrak{A}$, and
 - 1.3 each $\bigcup_{\beta < \alpha} M_\beta$ is a **finite** union $\bigcup_{i < m_\alpha} N_\alpha^i$ where $N_\alpha^i \prec \mathfrak{A}$.
2. Ignore M_α if ${}^2F \cap M_\alpha \subseteq \bigcup_{i < m_\alpha} N_\alpha^i$.
3. Otherwise, let ${}^2F \cap M_\alpha \setminus \bigcup_{i < m_\alpha} N_\alpha^i = \{p_k : k < \omega\}$.
4. For each p_k and N_α^i , there is at most one $n(k, i)$ such that the line through p_k parallel to $L_{n(k, i)}$ intersects N_α^i .
5. Put p_k in $S_{r(k)}$ where $r(k) \neq n(k, i)$ for all $i < m_\alpha$ and $r(k) \neq r(j)$ for all $j < k$.

The tree

To achieve the finite union of 1.3, Davies constructs \vec{M} as the leaves of what I will call a **Davies tree**.

- ▶ A Davies tree is of the form $(\mathfrak{A}_t : t \in T)$ where T is a well-founded tree of finite sequences of ordinals.
- ▶ The lexicographic ordering of the leaves of any such T is necessarily a well-ordering.
- ▶ ${}^2F \subseteq \mathfrak{A}_\emptyset \prec \mathfrak{A}$.
- ▶ If \mathfrak{A}_t is countable, then t is a leaf of T .
- ▶ If \mathfrak{A}_t is uncountable, then
 - ▶ $\mathfrak{A}_t = \bigcup \{ \mathfrak{A}_{t \frown \alpha} : t \frown \alpha \in T \}$,
 - ▶ $\alpha < \beta \Rightarrow \mathfrak{A}_{t \frown \alpha} \subseteq \mathfrak{A}_{t \frown \beta}$,
 - ▶ $\mathfrak{A}_{t \frown \alpha} \prec \mathfrak{A}_t$, and
 - ▶ $|\mathfrak{A}_{t \frown \alpha}| < |\mathfrak{A}_t|$.
- ▶ T is well-founded because $s \subset t$ implies $|\mathfrak{A}_s| > |\mathfrak{A}_t|$.

The finite union

- ▶ Let $t = (\alpha_0, \dots, \alpha_{m_\alpha-1})$ be a leaf of T .
- ▶ Every leaf $s <_{\text{lex}} t$ is of the form $(\alpha_0, \dots, \alpha_{i-1}, \beta, \gamma_{i+1}, \dots, \gamma_{n-1})$ where $i < m_\alpha$ and $\beta < \alpha_i$.
- ▶ Set $p(t, i, \beta) = (\alpha_0, \dots, \alpha_{i-1}, \beta)$.
- ▶ Set $\mathfrak{B}_{t,i} = \bigcup_{\beta < \alpha_i} \mathfrak{A}_{p(t,i,\beta)}$.
- ▶ $\bigcup_{s <_{\text{lex}} t} \mathfrak{A}_s = \bigcup_{i < m_\alpha} \mathfrak{B}_{t,i}$.
- ▶ $\mathfrak{B}_{t,i} \prec \mathfrak{A}$ or $\mathfrak{B}_{t,i} = \emptyset$.

Costs and benefits

Benefits

- ▶ We can construct something arbitrarily large one countable piece at a time: $(S_n \cap M_\alpha : n < \omega)$ for $\alpha < \kappa$.
- ▶ Our work done prior to handling M_α is a finite union of very nice pieces: $(S_n \cap N_\alpha^i : n < \omega)$ for $i < m_\alpha$.

Unavoidable Costs

- ▶ The nice pieces might interact in nasty ways because generally $N_\alpha^i \not\subseteq N_\alpha^j$.
- ▶ Davies trees only work in contexts where these interactions are sufficiently benign.

Avoidable costs

- ▶ “ $(N_\alpha^i : i < m_\alpha) \in M_\alpha$ ” is easy to arrange, but “ $(S_n \cap N_\alpha^i : i < m_\alpha \wedge n < \omega) \in M_\alpha$ ” is not.
- ▶ This could be a problem in some contexts.
- ▶ The work-around is to build \vec{S} and the Davies tree simultaneously...

Long ω_1 -approximation sequences

(M., 2008) There is a simpler structure that induces a Davies tree.

- ▶ Let \mathcal{L} be a countable language extending $\{\in\}$.
- ▶ Let \mathfrak{A} be an \mathcal{L} -expansion of $(H(\theta), \in)$.
- ▶ Let $(M_\alpha : \alpha < \eta)$ be a **long ω_1 -approximation sequence**:
 - ▶ M_α is countable and $M_\alpha \prec \mathfrak{A}$.
 - ▶ $(M_\beta : \beta < \alpha) \in M_\alpha$.
- ▶ It is easy to build \vec{M} and \vec{S} simultaneously.
- ▶ Warning: If $\alpha \geq \omega_1$, then $\alpha \not\subseteq M_\alpha$ and $\exists \beta < \alpha \ M_\beta \not\subseteq M_\alpha \wedge M_\beta \not\subseteq M_\alpha$.
- ▶ There is \emptyset -definable well-founded class tree Υ of finite sequences of ordinals such that if the first η leaves of Υ , according to the lexicographic ordering, are $(u_\alpha : \alpha < \eta)$, then $(\mathfrak{A}_t : t \in T)$ is a Davies tree where:
 - ▶ $T = \{t : \exists \alpha < \eta \ t \subseteq u_\alpha\}$.
 - ▶ $\mathfrak{A}_{u_\alpha} = M_\alpha$.
 - ▶ $\mathfrak{A}_t = \bigcup \{\mathfrak{A}_{t \frown \alpha} : t \frown \alpha \in T\}$ if t is not a leaf.

Ordinal division

- ▶ $\forall \alpha, \beta > 0 \exists! \gamma, \delta \quad \alpha = \beta \cdot \gamma + \delta \wedge \delta < \beta.$
- ▶ $\forall \alpha > 0 \exists! Q\alpha, R\alpha \quad \alpha = |\alpha| \cdot Q\alpha + R\alpha \wedge R\alpha < |\alpha|.$
- ▶ Observe that $1 \leq Q\alpha < \alpha^+$.

Repeatedly divide the remainder by its cardinality and call the result the **cardinal normal form** of α :

$$\alpha = |\alpha| \cdot Q\alpha + R\alpha$$

$$\alpha = |\alpha| \cdot Q\alpha + |R\alpha| \cdot QR\alpha + R^2\alpha$$

\vdots

$$\alpha = |\alpha| \cdot Q\alpha + |R\alpha| \cdot QR\alpha + |R^2\alpha| \cdot QR^2\alpha + \cdots + R^{m_\alpha}\alpha$$

Stop when $R^{m_\alpha}\alpha$ is countable.

Edge cases: $R^0 = \text{id}$; $m_\alpha = 0$ for all $\alpha < \omega_1$.

A canonical Davies tree

- ▶ The non-leaves of Υ are the sequences $(\tau_\alpha^j : j < i)$, with $i \leq m_\alpha$, of **truncations** $\tau_\alpha^j = \sum_{k < j} |R^k \alpha| \cdot QR^k \alpha$.
- ▶ For convenience, set $\tau^{m_\alpha} \alpha = \alpha$.
- ▶ The leaves are the sequences $(\tau_\alpha^j : j \leq m_\alpha)$.

Actually, we don't strictly need the tree anymore:

Given a long ω_1 -approximation sequence $(M_\alpha : \alpha < \eta)$,

- ▶ the **eras** of α are the intervals $I_\alpha^i = [\tau_\alpha^i, \tau_\alpha^{i+1})$ where $i < m_\alpha$;
- ▶ the **strata** of M_α are the unions $N_\alpha^i = \bigcup \{M_\beta : \beta \in I_\alpha^i\}$;
- ▶ $\bigcup_{\beta < \alpha} M_\beta = \bigcup_{i < m_\alpha} N_\alpha^i$;
- ▶ $N_\alpha^i \in M_\alpha$ and $|N_\alpha^i| \subseteq N_\alpha^i \prec \mathfrak{A}$;
- ▶ $i < j < m_\alpha \Rightarrow N_\alpha^i \in N_\alpha^j \wedge |N_\alpha^i| > |N_\alpha^j|$;
- ▶ N_α^i is uncountable, except possibly when $i = m_\alpha - 1$.

Another application of Davies trees

Theorem (Jackson and Mauldin, 2002)

There exists $S \subseteq {}^2\mathbb{R}$ such that $|S \cap L| = 1$ for every lattice L isometric with ${}^2\mathbb{Z}$.

- ▶ Jackson and Mauldin explicitly use a Davies tree in order to proceed one countable piece at a time, organizing all prior work into finitely many nice pieces.
- ▶ Proving that lattices $L_0 \in N_\alpha^0, \dots, L_{m_\alpha-1} \in N_\alpha^{m_\alpha-1}, L_{m_\alpha} \in M_\alpha$ interact sufficiently benignly takes many pages of work.

An implicit application to group theory

Theorem (Shelah, 1975)

Let G be a group and λ a singular cardinal such that every subgroup of G of size less than λ is free. Every subgroup of G of size λ is then also free.

- ▶ The above theorem is just a special case of Shelah's compactness theorem for singular cardinals.
- ▶ Shelah doesn't explicitly use a Davies tree, but he implicitly uses the first three non-root levels of a Davies tree, avoiding higher levels through an intricate inductive argument.

The κ -Freese Nation property

Fuchino, Koppelberg, Shelah (1996):

- ▶ Let κ be a regular cardinal.
- ▶ A boolean algebra A has the κ -**FN** if there is a map $f: A \rightarrow [A]^{<\kappa}$ such that for all $a \leq b$ we have $f(a) \cap f(b) \cap [a, b] \neq \emptyset$.
- ▶ **Classic example:** free booleans algebras have the \aleph_0 -FN.
- ▶ A subalgebra B of A is a κ -**subalgebra** of A , written $B \leq_\kappa A$, if every ideal of B of the form $B \cap [0, a]$ where $a \in A$ is generated by fewer than κ -many elements of B .
- ▶ **Small substructure characterization:** A has the κ -FN iff, for some club $\mathcal{E} \subseteq [A]^{<\kappa^+}$, every $B \in \mathcal{E}$ satisfies $B \leq_\kappa A$,

Using long κ^+ -approximation sequences, a natural generalization of long ω_1 -approximation sequences, I proved:

- ▶ **Large substructure characterization:** A has the κ -FN iff, for all $M \prec (H(\theta), \in, \leq_A)$ satisfying $\kappa^+ \cap M \in \kappa^+ + 1$, we have $A \cap M \leq_\kappa A$.

Openly generated compacta

- ▶ Let X be a compact (Hausdorff) space.
- ▶ Let $C(X)$ be the algebra of all continuous $f: X \rightarrow \mathbb{R}$.
- ▶ Given a class \mathcal{A} , let X/\mathcal{A} denote the quotient space where points are identified iff no $f \in C(X) \cap \mathcal{A}$ distinguishes them.
- ▶ **Small quotient characterization (Ščepin, 1981):** X is **openly generated** iff, for some club $\mathcal{E} \subseteq [C(X)]^{<\aleph_1}$, the quotient map $q_M^X: X \rightarrow X/M$ is open for all $M \in \mathcal{E}$.
- ▶ **Large quotient characterization (M., 2008):** X is openly generated iff, for all $M \prec (H(\theta), \in, C(X))$, the quotient map $q_M^X: X \rightarrow X/M$ is open.
- ▶ **Algebraic connection (classical):** If X is zero-dimensional, then X is openly generated iff $\text{Clop}(X)$ has the \aleph_0 -FN.
- ▶ **Example (easy):** Powers of 2 are openly generated.
- ▶ **Example (Šapiro, 1976):** The Vietoris hyperspace of ${}^{\aleph_2}2$ is openly generated but not a continuous image of a power of 2.

Flat bases

- ▶ A (local) base of a space is **flat** if every element of the (local) base has only finitely many supersets in the (local) base.
- ▶ (M., 2008) If Y is a continuous image of an openly generated compact X , and \mathcal{A} is a local base in Y , then \mathcal{A} contains a flat local base \mathcal{B} in Y .
 - ▶ This result didn't need a long ω_1 -approximation sequence for its proof, just a continuous elementary \in -chain.
- ▶ A space Y is **homogeneous** if for all $p, q \in Y$, there is a homeomorphism $f: Y \rightarrow Y$ such that $f(p) = q$.
- ▶ (M., 2008) If Y is a homogeneous continuous image of an openly generated compact X , and \mathcal{A} is a base of Y , then \mathcal{A} contains a flat base \mathcal{B} of Y .
 - ▶ The result was proved with a long ω_1 -approximation sequence.
 - ▶ A stronger result for metrizable compacta was used to ensure that the inductive construction succeeded.
 - ▶ The crucial inductive argument is that if each $\mathcal{B} \cap N_\alpha^i$ is flat, then, for all nonempty open U , $q_{N_\alpha^i}^Y[U]$ has only finitely many supersets in $\mathcal{B} \cap \bigcup_{\beta < \alpha} M_\alpha$.

Why flatness?

(M., 2008) Every known homogeneous compact space (including all compact groups) has a flat local base.
(By homogeneity, this is equivalent to saying that every local base contains a flat local base.)

Conjecture. Every homogeneous compact space has a flat local base.

Why homogeneity?

- ▶ A (local) base is κ -flat if every element of the (local) base has fewer than κ -many supersets in the (local) base.
- ▶ (M., 2008) Every known homogeneous compact space has a \mathfrak{c}^+ -flat base.
- ▶ The **cellularity** $c(X)$ of a space X is the supremum of the cardinalities of its pairwise disjoint families of open sets.
- ▶ **Van Douwen's Problem (c. 1970)**. Is there a homogeneous compact space with cellularity greater than \mathfrak{c} ?
- ▶ Van Douwen's Problem is still open in all models of ZFC.
- ▶ (M., 2008) If GCH holds, then, for all homogeneous compact X , every local base in X contains a $c(X)$ -flat local base.
- ▶ This is weak evidence for “no” to Van Douwen. Nevertheless...
- ▶ **Conjecture**. There are homogeneous compact X with $c(X) > \mathfrak{c}$ because every compact Y is a continuous image of a homogeneous compact X .

A new application to homogeneous compacta

(M., 2012) If W is a 0-dimensional openly generated compactum, then W is a continuous open image of a homogeneous openly generated compactum Z .

- ▶ The result is proved using long ω_1 -approximation sequences.
- ▶ The continuous surjection $g: Z \rightarrow W$ is built using inverse limits of maps $f_{M_\alpha}: X_{M_\alpha} \rightarrow Y/M_\alpha$ with bonding maps $\pi_\alpha^i: X_{M_\alpha} \rightarrow X_{N_\alpha^i}/M_\alpha$ such that diagrams of the following form commute:

$$\begin{array}{ccc} X_{M_\alpha} & \xrightarrow{f_{M_\alpha}} & Y/M_\alpha \\ \pi_\alpha^i \downarrow & & \downarrow q_{N_\alpha^i}^{Y/M_\alpha} \\ X_{N_\alpha^i}/M_\alpha & \xrightarrow{f_{N_\alpha^i}/M_\alpha} & Y/(N_\alpha^i \cap M_\alpha) \end{array}$$

- ▶ Here $X_{N_\alpha^i}$ and $f_{N_\alpha^i}$ are themselves inverse limits of $(X_{M_\beta} : \beta \in I_\alpha^i)$ and $(f_{M_\beta} : \beta \in I_\alpha^i)$.
- ▶ That $N_\alpha^i = \bigcup \{M_\beta : \beta \in I_\alpha^i\}$ is a directed union is proved by induction on the Davies tree.