PHYS 2325: 2D COLLISION

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Question. Imagine discs sliding on a flat sheet of ice. Disc A, with mass $m_A = 0.35kg$ and initial velocity $\vec{\boldsymbol{v}}_A = 0.75\hat{\boldsymbol{\imath}}(m/s)$, hits disc B, which has mass $m_B = 0.95kg$ and initial speed $v_B = 0.00m/s$. After the collision, disc B has final velocity $\vec{\boldsymbol{v}}^*_B$ directed $\theta = 42°$ clockwise from $\hat{\boldsymbol{\jmath}}$. For simplicity, assume that the collision is elastic and that friction (and all other external forces) and spin are negligible. What are the final velocities $\vec{\boldsymbol{v}}^*_A$ and $\vec{\boldsymbol{v}}^*_B$?

Answer. By definition of elastic collision, the total kinetic energy is conserved:

$$\frac{1}{2}m_A(v^*_A)^2 + \frac{1}{2}m_B(v^*_B)^2 = K^*_A + K^*_B = K_A + K_B = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2.$$

Since all external forces are negligible, the total momentum is conserved:

$$m_A\vec{\boldsymbol{v}}^*_A + m_B\vec{\boldsymbol{v}}^*_B = \vec{\boldsymbol{p}}^*_A + \vec{\boldsymbol{p}}^*_B = \vec{\boldsymbol{p}}_A + \vec{\boldsymbol{p}}_B = m_A\vec{\boldsymbol{v}}_A + m_B\vec{\boldsymbol{v}}_B.$$

Break up this vector equation into x and y components.

$$\begin{cases} m_A v^*_{Ax} + m_B v^*_{Bx} & = p^*_{Ax} + p^*_{Bx} & = p_{Ax} + p_{Bx} & = m_A v_{Ax} + m_B v_{Bx} \\ m_A v^*_{Ay} + m_B v^*_{By} & = p^*_{Ax} + p^*_{Bx} & = p_{Ax} + p_{Bx} & = m_A v_{Ay} + m_B v_{By} \end{cases}$$

Thus, we have three equations and apparently six unknowns, $v^*_A, v^*_{Ax}, v^*_{Ay}, v^*_B, v^*_{Bx}, v^*_{By}$. However, we can use θ and a little geometry get the remaining equations we need. By elementary trigonometry, $v^*_{Bx} = v^*_B \sin\theta$ and $v^*_{By} = v^*_B \cos\theta$. Also, note that $v_{Ax} = v_A$, $v_{Ay} = 0$, and $v_{Bx} = v_{By} = v_B = 0$. Our three equations now simplify:

$$\begin{cases} \frac{1}{2}m_A(v^*_A)^2 + \frac{1}{2}m_B(v^*_B)^2 & = \frac{1}{2}m_A v_A^2 \\ m_A v^*_{Ax} + m_B v^*_B \sin\theta & = m_A v_A \\ m_A v^*_{Ay} + m_B v^*_B \cos\theta & = 0 \end{cases}$$

Solve the last two equations for v^*_{Ax} and v^*_{Ay}:

$$\begin{cases} v^*_{Ax} & = v_A - m_A^{-1} m_B v^*_B \sin\theta \\ v^*_{Ay} & = -m_A^{-1} m_B v^*_B \cos\theta \end{cases}$$

Apply the Pythagorean Theorem to $\vec{\boldsymbol{v}}^*_A$:

$$\begin{aligned} (v^*_A)^2 &= (v^*_{Ax})^2 + (v^*_{Ay})^2 \\ &= (v_A - m_A^{-1} m_B v^*_B \sin\theta)^2 + (-m_A^{-1} m_B v^*_B \cos\theta)^2 \\ &= v_A^2 - 2v_A m_A^{-1} m_B v^*_B \sin\theta + m_A^{-2} m_B^2 (v^*_B)^2 \sin^2\theta + m_A^{-2} m_B^2 (v^*_B)^2 \cos^2\theta \\ &= v_A^2 - 2v_A m_A^{-1} m_B v^*_B \sin\theta + m_A^{-2} m_B^2 (v^*_B)^2 \end{aligned}$$

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Substitute this formula for $(v_A^*)^2$ into our simplified equation for the conservation of kinetic energy and solve for v_B^*:

$$\begin{aligned}
\tfrac{1}{2}m_A v_A^2 &= \tfrac{1}{2}m_A(v_A^*)^2 + \tfrac{1}{2}m_B(v_B^*)^2 \\
\tfrac{1}{2}m_A v_A^2 &= \tfrac{1}{2}m_A(v_A^2 - 2v_A m_A^{-1} m_B v_B^* \sin\theta + m_A^{-2} m_B^2 (v_B^*)^2) + \tfrac{1}{2}m_B(v_B^*)^2 \\
m_A v_A^2 &= m_A(v_A^2 - 2v_A m_A^{-1} m_B v_B^* \sin\theta + m_A^{-2} m_B^2 (v_B^*)^2) + m_B(v_B^*)^2 \\
m_A v_A^2 &= m_A v_A^2 - 2v_A m_B v_B^* \sin\theta + m_A^{-1} m_B^2 (v_B^*)^2 + m_B(v_B^*)^2 \\
0 &= -2v_A m_B v_B^* \sin\theta + m_A^{-1} m_B^2 (v_B^*)^2 + m_B(v_B^*)^2 \\
0 &= v_B^*(-2v_A m_B \sin\theta + m_A^{-1} m_B^2 v_B^* + m_B v_B^*)
\end{aligned}$$

The product of two quantities is zero exactly when one (or both) of the factors is zero. The first possibility, $v_B^* = 0$, corresponds to the physical situation in which disc A merely passes by disc B without hitting it. By assumption, disc A actually hits disc B, so $v_B^* \neq 0$ and the other factor is zero:

$$\begin{aligned}
0 &= -2v_A m_B \sin\theta + m_A^{-1} m_B^2 v_B^* + m_B v_B^* \\
0 &= -2v_A m_B \sin\theta + (m_A^{-1} m_B^2 + m_B) v_B^* \\
2v_A m_B \sin\theta &= (m_A^{-1} m_B^2 + m_B) v_B^* \\
2v_A m_A m_B \sin\theta &= (m_B^2 + m_A m_B) v_B^* \\
\tfrac{2v_A m_A m_B \sin\theta}{m_B^2 + m_A m_B} &= v_B^* \\
\tfrac{2v_A m_A \sin\theta}{m_B + m_A} &= v_B^*
\end{aligned}$$

Now plug our solution for v_B^* into v_{Ax}^*, v_{Ay}^*, v_{Bx}^*, and v_{By}^*. As an optional step, we can simplify using the double-angle formulas $\cos 2\theta = 1 - 2\sin^2\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$.

$$\begin{cases}
v_{Ax}^* = v_A - m_A^{-1} m_B v_B^* \sin\theta = v_A - \frac{2v_A m_B \sin^2\theta}{m_B + m_A} = \frac{v_A(m_B + m_A)}{m_B + m_A} - \frac{2v_A m_B \sin^2\theta}{m_B + m_A} = \frac{v_A(m_A + m_B \cos 2\theta)}{m_A + m_B} \\
v_{Ay}^* = -m_A^{-1} m_B v_B^* \cos\theta = -\frac{2v_A m_B \sin\theta\cos\theta}{m_B + m_A} = -\frac{v_A m_B \sin 2\theta}{m_A + m_B} \\
v_{Bx}^* = v_B^* \sin\theta = \frac{2v_A m_A \sin^2\theta}{m_A + m_B} = \frac{v_A m_A (1 - \cos 2\theta)}{m_A + m_B} \\
v_{By}^* = v_B^* \cos\theta = \frac{2v_A m_A \sin\theta\cos\theta}{m_B + m_A} = \frac{v_A m_A \sin 2\theta}{m_A + m_B}
\end{cases}$$

Plugging in our given values for m_A, m_B, v_A, θ into our above solutions, we get:

$$\begin{aligned}
\vec{\boldsymbol{v}}_A^* &= v_{Ax}^* \hat{\boldsymbol{\imath}} + v_{Ay}^* \hat{\boldsymbol{\jmath}} &&= (0.259213\hat{\boldsymbol{\imath}} - 0.545075\hat{\boldsymbol{\jmath}})(m/s) \\
\vec{\boldsymbol{v}}_B^* &= v_{Bx}^* \hat{\boldsymbol{\imath}} + v_{By}^* \hat{\boldsymbol{\jmath}} &&= (0.180816\hat{\boldsymbol{\imath}} + 0.200817\hat{\boldsymbol{\jmath}})(m/s)
\end{aligned}$$

Rounding to 2 significant figures, our final answers are $\vec{\boldsymbol{v}}_A^* = (0.26\hat{\boldsymbol{\imath}} - 0.55\hat{\boldsymbol{\jmath}})(m/s)$ and $\vec{\boldsymbol{v}}_B^* = (0.18\hat{\boldsymbol{\imath}} + 0.20\hat{\boldsymbol{\jmath}})(m/s)$.

Confirmation. Let us numerically test our answers. The momenta below are in meters per second; the kinetic energies are in Joules.

$$\begin{aligned}
m_A v_{Ax} + m_B v_{Bx} &= (0.35)(0.75) + (0.95)(0) = 0.2625 \\
m_A v^*_{Ax} + m_B v^*_{Bx} &= (0.35)(0.259213) + (0.95)(0.180816) = 0.262500
\end{aligned}$$

$$\begin{aligned}
m_A v_{Ay} + m_B v_{By} &= (0.35)(0) + (0.95)(0) = 0 \\
m_A v^*_{Ay} + m_B v^*_{By} &= (0.35)(-0.545075) + (0.95)(0.200817) = 0.000000
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2 &= (0.5)(0.35)(0.5626) + (0.5)(0.95)(0) = 0.0984375 \\
\tfrac{1}{2} m_A (v^*_A)^2 + \tfrac{1}{2} m_B (v^*_B)^2 &= \tfrac{1}{2} m_A ((v^*_{Ax})^2 + (v^*_{Ay})^2) + \tfrac{1}{2} m_B ((v^*_{Bx})^2 + (v^*_{By})^2) \\
&= (0.5)(0.35)(0.0671912 + 0.297106) + (0.5)(0.95)(0.0326946 + 0.0403274) \\
&= 0.0984375
\end{aligned}$$