ESTIMATING SUMS WITH INTEGRALS

Suppose you want to estimate $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots$. From the integral test,

$$\int_1^\infty \frac{dx}{x^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq \int_1^\infty \frac{dx}{x^3} + \frac{1}{1^3}.$$

We evaluate the above improper integral and see that it converges to 1/2:

$$\int_1^\infty x^{-3}\,dx = \lim_{t\to\infty} \int_1^t x^{-3}\,dx = \lim_{t\to\infty} \left(\frac{x^{-2}}{-2}\bigg|_1^t \right) = \lim_{t\to\infty} \left(\frac{t^{-2}}{-2} - \frac{1^{-2}}{-2} \right) = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) = \frac{1}{2}.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, and it converges to something between $\frac{1}{2}$ and $\frac{1}{2} + \frac{1}{1^3} = \frac{3}{2}$.

We can get a much better estimate as follows. First, evaluate part of $\sum_{n=1}^{\infty} \frac{1}{n^3}$, say, the sum of the first 9 terms, $\sum_{n=1}^{9} \frac{1}{n^3}$, by simply adding them together:

$$\sum_{n=1}^{9} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{9^3} \approx 1.196531986.$$

Second, estimate the rest of the sum, $\sum_{n=10}^{\infty} \frac{1}{n^3} = \frac{1}{10^3} + \frac{1}{11^3} + \frac{1}{12^3} + \cdots$, using integral test estimates:

$$\int_{10}^\infty \frac{dx}{x^3} \leq \sum_{n=10}^{\infty} \frac{1}{n^3} \leq \int_{10}^\infty \frac{dx}{x^3} + \frac{1}{10^3}$$

$$\int_{10}^\infty \frac{dx}{x^3} = \lim_{t\to\infty} \left(\frac{x^{-2}}{-2}\bigg|_{10}^t \right) = \lim_{t\to\infty} \left(\frac{t^{-2}}{-2} - \frac{10^{-2}}{-2} \right) = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{200} \right) = \frac{1}{200}.$$

$$.005 = \frac{1}{200} = \int_{10}^\infty \frac{dx}{x^3} \leq \sum_{n=10}^{\infty} \frac{1}{n^3} \leq \int_{10}^\infty \frac{dx}{x^3} + \frac{1}{10^3} = \frac{1}{200} + \frac{1}{1000} = .006$$

Finally, we add our value for $\sum_{n=1}^{9} \frac{1}{n^3}$ to our estimate of $\sum_{n=10}^{\infty} \frac{1}{n^3}$ to get an estimate of $\sum_{n=1}^{\infty} \frac{1}{n^3}$:

$$1.196531986 + .005 \approx \sum_{n=1}^{9} \frac{1}{n^3} + \int_{10}^\infty \frac{dx}{x^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq \sum_{n=1}^{9} \frac{1}{n^3} + \int_{10}^\infty \frac{dx}{x^3} + \frac{1}{10^3} \approx 1.196531986 + .006$$

$$1.201531986 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1.202531986.$$

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My computer (using a more advanced estimation technique) says that $\sum\limits_{n=1}^{\infty} \frac{1}{n^3} = 1.2020569032\ldots$ (correct to ten decimal places), in agreement with our estimate.

In general, the integral test tells us that if $p > 1$, then

$$\frac{1}{p-1} = \int_1^{\infty} \frac{dx}{x^p} \leq \sum_{n=1}^{\infty} \frac{1}{n^p} \leq \int_1^{\infty} \frac{dx}{x^p} + \frac{1}{1^p} = \frac{p}{p-1}.$$

To get a better a estimate, we pick a number K, compute the first $K-1$ terms of the sum directly, and then estimate the rest using the integral test:

$$\sum_{n=1}^{K-1} \frac{1}{n^p} + \int_K^{\infty} \frac{dx}{x^p} \leq \sum_{n=1}^{\infty} \frac{1}{n^p} \leq \sum_{n=1}^{K-1} \frac{1}{n^p} + \int_K^{\infty} \frac{dx}{x^p} + \frac{1}{K^p}$$

$$\sum_{n=1}^{K-1} \frac{1}{n^p} + \frac{1}{(p-1)K^{p-1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^p} \leq \sum_{n=1}^{K-1} \frac{1}{n^p} + \frac{1}{(p-1)K^{p-1}} + \frac{1}{K^p}.$$

The difference between our lower and upper bounds is $\frac{1}{K^p}$, so by adding up the first $K-1$ terms, and then adding the appropriate integral, we get an estimate with an error less than $\frac{1}{K^p}$.

The formula $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is important enough to have its own abbreviation, $\zeta(p)$ (pronounced "zeta of p"). There's a \$1,000,000 prize for anyone who can answer a certain famous question about the ζ function. Google "Riemann Hypothesis."