Calculus II (2414): AN INTEGRAL FOR FEB. 11

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Let us compute $\int_3^4 \frac{dx}{\sqrt{x^2-2x+5}}$.

Complete the square using $x^2+px = \left(x+\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$.

$$x^2-2x+5 = \left(x-\frac{2}{2}\right)^2 - \left(-\frac{2}{2}\right)^2 + 5 = (x-1)^2 - 1 + 5 = (x-1)^2+4$$

$$\int_3^4 \frac{dx}{\sqrt{x^2-2x+5}} = \int_3^4 \frac{dx}{\sqrt{(x-1)^2+4}}$$

Substitute $u = x-1$. We have $du/dx = 1$, so $du = dx$. The u-limits are $4-1$ and $3-1$, that is, 3 and 2.

$$\int_3^4 \frac{dx}{\sqrt{(x-1)^2+4}} = \int_2^3 \frac{du}{\sqrt{u^2+4}}$$

Substitute $u = 2\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We have $du/d\theta = 2\sec^2\theta$, so $du = 2\sec^2\theta\, d\theta$. Moreover, $\tan\theta = u/2$, so $\theta = \tan^{-1}(u/2)$, so the θ-limits are $\tan^{-1}(3/2)$ and $\tan^{-1}1$. Finally, $\sqrt{u^2+4} = \sqrt{4\tan^2\theta+4} = 2\sqrt{\tan^2\theta+1} = 2\sec\theta$. Therefore,

$$\int_2^3 \frac{du}{\sqrt{u^2+4}} = \int_{\tan^{-1}1}^{\tan^{-1}\frac{3}{2}} \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int_{\tan^{-1}1}^{\tan^{-1}\frac{3}{2}} \sec\theta\, d\theta = \left(\ln|\sec\theta+\tan\theta|\right)_{\tan^{-1}1}^{\tan^{-1}\frac{3}{2}}.$$

Thus,

$$\int_3^4 \frac{dx}{\sqrt{x^2-2x+5}} = \ln\left|\sec\left(\tan^{-1}\frac{3}{2}\right) + \tan\left(\tan^{-1}\frac{3}{2}\right)\right| - \ln\left|\sec\left(\tan^{-1}1\right) + \tan\left(\tan^{-1}1\right)\right|$$

By definition, $\tan(\tan^{-1}x) = x$ for all x, but how do we compute $\sec(\tan^{-1}x)$ for a given x? By definition, any angle of the form $\theta = \tan^{-1}x$ satisfies $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so $\sec\theta > 0$, which implies $\sec\theta = \sqrt{1+\tan^2\theta}$. Therefore, $\sec(\tan^{-1}x) = \sqrt{1+\tan^2(\tan^{-1}x)} = \sqrt{1+x^2}$. Hence,

$$\int_3^4 \frac{dx}{\sqrt{x^2-2x+5}} = \ln\left|\sqrt{1+\left(\frac{3}{2}\right)^2} + \frac{3}{2}\right| - \ln\left|\sqrt{1+1^2}+1\right|$$

Let's simplify our answer.

$$\ln\left|\sqrt{1+\left(\frac{3}{2}\right)^2} + \frac{3}{2}\right| - \ln\left|\sqrt{1+1^2}+1\right| = \ln\left|\frac{\sqrt{1+\left(\frac{3}{2}\right)^2}+\frac{3}{2}}{\sqrt{1+1^2}+1}\right| = \ln\left|\frac{\sqrt{\frac{13}{4}}+\frac{3}{2}}{\sqrt{2}+1}\right| = \ln\left|\frac{\sqrt{13}+3}{2(\sqrt{2}+1)}\right|$$