

# MORE ABOUT NOETHERIAN TYPES OF COMPACTA

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ABSTRACT. A compact LOTS has an  $\omega_1^{\text{op}}$ -like base if and only if it is metrizable. An infinite cardinal  $\kappa$  is the Noetherian type of a compact LOTS if and only if  $\kappa \neq \omega_1$  and  $\kappa$  is not weakly inaccessible. If  $X$  is a homogeneous compactum with regular weight, then every base of  $X$  contains an  $Nt(X)^{\text{op}}$ -like base.

## 1. LINDELÖF LINEARLY ORDERED TOPOLOGICAL SPACES

We will show that a Lindelöf linearly ordered topological space has an  $\omega^{\text{op}}$ -like base if and only if it is metric. Moreover, a compact linearly ordered topological space has an  $\omega_1^{\text{op}}$ -like base if and only if it is metric.

**Theorem 1.1.** *Every metric space has an  $\omega^{\text{op}}$ -like base.*

*Proof.* Let  $X$  be a metric space. For each  $n < \omega$ , let  $\mathcal{A}_n$  be a locally finite open refinement of the balls of radius  $2^{-n}$  in  $X$ . Set  $\mathcal{A} = \bigcup_{n < \omega} \mathcal{A}_n$ . Then  $\mathcal{A}$  is a base of  $X$  because if  $p \in X$  and  $n < \omega$ , then there exists  $U \in \mathcal{A}_{n+1}$  such that  $p \in U$  and  $U$  is contained in the ball of radius  $2^{-n}$  with center  $p$ . Let us show that  $\mathcal{A}$  is  $\omega^{\text{op}}$ -like. Suppose  $m < \omega$  and  $U \in \mathcal{A}$  and  $V \in \mathcal{A}_m$  and  $U \subsetneq V$ . Then there exist  $p \in U$  and  $\epsilon_0 > \epsilon_1 > 0$  such that the  $\epsilon_0$ -ball with center  $p$  is contained in  $U$  and the  $\epsilon_1$ -ball with center  $p$  intersects only finitely many elements of  $\mathcal{A}_n$  for all  $n < \omega$  satisfying  $2^{-n} > \epsilon_0/2$ . If  $2^{-m} \leq \epsilon_0/2$ , then  $V$  is contained in the  $\epsilon_0$ -ball with center  $p$ , in contradiction with  $U \subsetneq V$ . Hence,  $2^{-m} > \epsilon_0/2$ ; hence, there are only finitely many possibilities for  $m$  and  $V$  given  $U$ , for  $V$  intersects the  $\epsilon_1$ -ball with center  $p$ .  $\square$

**Lemma 1.2.** *Let  $X$  be a Lindelöf linearly ordered topological space with open cover  $\mathcal{A}$ . Then  $\mathcal{A}$  has a countable, locally finite refinement consisting only of countable unions of open intervals.*

*Proof.* Let  $\{A_n : n < \omega\}$  be a countable refinement of  $\mathcal{A}$  consisting only of open intervals. For each  $n < \omega$ , set  $B_n = A_n \setminus \bigcup_{m < n} A_m$ ; set  $\mathcal{B} = \{B_n : n < \omega\}$ . Then  $\mathcal{B}$  is a locally finite refinement of  $\mathcal{A}$ . Let  $\mathcal{C}$  be the set of open intervals of  $X$  which intersect only finitely many elements of  $\mathcal{B}$ . Let  $\mathcal{D}$  be the set of  $U \in \mathcal{C}$  satisfying  $\overline{U} \subseteq V$  for some  $V \in \mathcal{C}$ . Let  $\{D_n : n < \omega\}$  be a countable subcover of  $\mathcal{D}$ . For each  $n < \omega$ , set  $E_n = \overline{D_n} \setminus \bigcup_{m < n} \overline{D_m}$ ; set  $\mathcal{E} = \{E_n : n < \omega\}$ . Then  $\mathcal{E}$  is a locally finite refinement of  $\mathcal{C}$ . For each  $n < \omega$ , set  $F_n = A_n \setminus \bigcup\{E \in \mathcal{E} : B_n \cap E = \emptyset\}$ , which is a countable union of intervals; set  $\mathcal{F} = \{F_n : n < \omega\}$ . Since  $\mathcal{E}$  is locally finite, each  $F_n$  is open. Hence, each  $F_n$  is a countable union of open intervals. Moreover,  $B_n \subseteq F_n \subseteq A_n$  for all  $n < \omega$ ; hence,  $\mathcal{F}$  is a refinement of  $\mathcal{A}$ .

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Thus, it suffices to show that  $\mathcal{F}$  is locally finite. Since  $\mathcal{E}$  is a locally finite cover of  $X$ , it suffices to show that each element of  $\mathcal{E}$  only intersects finitely many elements of  $\mathcal{F}$ . Let  $i < \omega$  and choose  $V \in \mathcal{C}$  such that  $E_i \subseteq V$ . Suppose  $j < \omega$  and  $E_i \cap F_j \neq \emptyset$ . Then  $E_i \cap B_j \neq \emptyset$  by definition of  $F_j$ . Hence,  $V \cap B_j \neq \emptyset$ ; hence, there are only finitely possibilities for  $B_j$ ; hence, there are only finitely many possibilities for  $F_j$ .  $\square$

**Lemma 1.3.** *Let  $X$  be a nonseparable, Lindelöf, linearly ordered topological space. Then  $X$  does not have an  $\omega^{\text{op}}$ -like base.*

*Proof.* Let  $\mathcal{A}$  be a base of  $X$ . Let us show that  $\mathcal{A}$  is not  $\omega^{\text{op}}$ -like. First, let us construct sequences of open sets  $\langle A_{n,k} \rangle_{n,k < \omega}$  and  $\langle B_{n,k} \rangle_{n,k < \omega}$ . Our requirements are that  $B_{n,i} \subseteq A_{n,i} \in \mathcal{A}$ , that  $B_{n,i}$  is a countable union of open intervals, that  $\{B_{n,k} : k < \omega\}$  is a locally finite cover of  $X$  and pairwise  $\subseteq$ -incomparable, and that  $\{A_{i,k} : k < \omega\} \cap \{A_{j,k} : k < \omega\} \subseteq [X]^1$  for all  $i < j < \omega$  and  $n < \omega$ .

Suppose  $n < \omega$  and we are given  $\langle A_{m,k} \rangle_{k < \omega}$  and  $\langle B_{m,k} \rangle_{k < \omega}$  for all  $m < n$  and they meet our requirements. Let  $p \in X$ . Set  $V_p = \bigcap \{B_{m,k} : m < n \text{ and } k < \omega \text{ and } p \in B_{m,k}\}$ . Then  $V_p$  is open. If  $|V_p| = 1$ , then set  $U_p = V_p$ . If  $|V_p| > 1$ , then choose  $U_p \in \mathcal{A}$  such that  $p \in U_p \subseteq V_p$ . Set  $\mathcal{U} = \{U_p : p \in X\}$ . By Lemma 1.2, there exists a countable, locally finite refinement  $\mathcal{B}_n$  of  $\mathcal{U}$  consisting only of countable unions of open intervals. Since  $\mathcal{B}_n$  is locally finite, it has no infinite ascending chains; hence, we may assume  $\mathcal{B}_n$  is pairwise  $\subseteq$ -incomparable because we may shrink  $\mathcal{B}_n$  to its maximal elements. Let  $\{B_{n,k} : k < \omega\} = \mathcal{B}_n$ . For each  $k < \omega$ , set  $A_{n,k} = U_p$  for some  $p \in X$  satisfying  $B_{n,k} \subseteq U_p$ . Suppose  $m < n$  and  $i, j < \omega$  and  $A_{m,i} = A_{n,j} \notin [X]^1$ . Choose  $p \in X$  such that  $A_{n,j} = U_p$ ; choose  $k < \omega$  such that  $p \in B_{m,k}$ . Then  $B_{m,i} \subseteq A_{m,i} = U_p \subseteq V_p \subseteq B_{m,k}$ , in contradiction with the pairwise  $\subseteq$ -incomparability of  $\{B_{m,l} : l < \omega\}$ . Thus,  $\{A_{m,l} : l < \omega\} \cap \{A_{n,l} : l < \omega\} \subseteq [X]^1$  for all  $m < n$ . By induction,  $\langle A_{n,k} \rangle_{n,k < \omega}$  and  $\langle B_{n,k} \rangle_{n,k < \omega}$  meet our requirements.

Let  $\{X, \leq, \mathcal{A}\} \subseteq M \prec H_\theta$  and  $|M| = \omega$ . Choose  $x \in X \setminus \overline{X \cap M}$ . Then there exists  $y, z \in X$  such that  $y < x < z$  and  $(y, z)$  does not intersect  $M$ . Choose  $U \in \mathcal{A}$  such that  $U \subseteq (y, z)$ . By elementarity, we may assume that  $A_{n,k}, B_{n,k} \in M$  for all  $n, k < \omega$ . For each  $n < \omega$ , choose  $i_n < \omega$  such that  $x \in B_{n,i_n}$ . Fix  $n < \omega$ . Since  $x \notin M$ , we cannot have  $A_{n,i_n} = \{x\}$ ; hence,  $A_{n,i_n} \neq A_{m,i_m}$  for all  $m < n$ . Hence, it suffices to show that  $U \subseteq A_{n,i_n}$ . There exist  $\langle u_j \rangle_{j < \omega}, \langle v_j \rangle_{j < \omega} \in (X \cup \{\infty, -\infty\})^\omega \cap M$  such that  $B_{n,i_n} = \bigcup_{j < \omega} (u_j, v_j)$ . Hence, there exists  $j < \omega$  such that  $u_j < x < v_j$ . Since  $x \in (y, z) \cap (u_j, v_j)$  and  $(y, z)$  does not intersect  $M$ , we have  $(y, z) \subseteq (u_j, v_j)$ ; hence,  $U \subseteq A_{n,i_n}$ .  $\square$

**Proposition 1.4.** *If  $X$  is a space and  $\pi(X) < \text{cf } \kappa \leq \kappa \leq w(X)$ , then  $Nt(X) > \kappa$ .*

*Proof.* Suppose  $\mathcal{A}$  is a base of  $X$  and  $\mathcal{B}$  is  $\pi$ -base of  $X$  of size  $\pi(X)$ . Then  $|\mathcal{A}| \geq \kappa$ ; hence, there exist  $\mathcal{U} \in [\mathcal{A}]^\kappa$  and  $V \in \mathcal{B}$  such that  $V \subseteq \bigcap \mathcal{U}$ . Hence, there exists  $W \in \mathcal{A}$  such that  $W \subseteq V \subseteq \bigcap \mathcal{U}$ ; hence,  $\mathcal{A}$  is not  $\kappa^{\text{op}}$ -like.  $\square$

**Theorem 1.5.** *Let  $X$  be a Lindelöf linearly ordered topological space. Then the following are equivalent.*

- (1)  $X$  is metric.
- (2)  $X$  has an  $\omega^{\text{op}}$ -like base.
- (3)  $X$  is separable and has an  $\omega_1^{\text{op}}$ -like base.

*Proof.* By Theorem 1.1, (1) implies (2). By Lemma 1.3, (2) implies (3). Hence, it suffices to show that (3) implies (1). Suppose  $X$  has a countable dense subset  $D$  and an  $\omega_1^{\text{op}}$ -like base. Then  $\pi(X) = \omega$ ; hence, by Proposition 1.4,  $w(X) = \omega$ ; hence,  $X$  is metric.  $\square$

For compact linearly ordered topological spaces, Theorem 1.5 can be strengthened.

**Lemma 1.6.** *Suppose  $\kappa$  is a regular uncountable cardinal and  $X$  is a linearly ordered compactum such that  $Nt(X) \leq \kappa$ . Then  $d(X) < \kappa$ .*

*Proof.* Suppose  $d(X) \geq \kappa$  and  $\mathcal{A}$  is a  $\kappa^{\text{op}}$ -like base of  $X$ . Let  $\{X, \leq, \mathcal{A}\} \in M \prec H_\theta$  and  $|M| < \kappa$  and  $M \cap \kappa \in \kappa$ . By compactness,  $X$  contains a nonempty open interval  $(x, y)$  that is maximal among the open convex subsets of  $X$  that are disjoint from  $M$ . If  $x, y \in M$ , then  $(x, y) \cap M$  is nonempty by elementarity; hence, we may assume  $x \notin M$ . Therefore, by maximality of  $(x, y)$ , we have  $x = \sup([\min X, x) \cap M)$ . Choose  $z \in (x, y)$ ; choose  $U \in \mathcal{A}$  such that  $x \in U \subseteq [\min X, z)$ . Then there exist  $u, v \in X$  such that  $x \in (u, v) \subseteq U$ . Hence, there exist  $p_0, p_1, p_2 \in M$  such that  $u < p_0 < p_1 < p_2 < x$ . Choose  $V \in \mathcal{A}$  such that  $p_1 \in V \subseteq (p_0, p_2)$ ; by elementarity, we may assume  $V \in M$ . Set  $\mathcal{B} = \{W \in \mathcal{A} : V \subseteq W\}$ . Then  $U \in \mathcal{B} \in M$  and  $|\mathcal{B}| < \kappa$ ; hence,  $U \in M$ . Set  $w = \min([p_1, \max X] \setminus U)$ . Then  $w \in M$  and  $v \leq w \leq z$ ; hence,  $w \in (x, y) \cap M$ , which is absurd. Thus,  $d(X) < \kappa$ .  $\square$

**Theorem 1.7.** *Let  $X$  be a linearly ordered compactum. Then the following are equivalent.*

- (1)  $X$  is metric.
- (2)  $X$  has an  $\omega^{\text{op}}$ -like base.
- (3)  $X$  has an  $\omega_1^{\text{op}}$ -like base.
- (4)  $X$  is separable and has an  $\omega_1^{\text{op}}$ -like base.

*Proof.* By Theorem 1.5, (1), (2), and (4) are equivalent. Moreover, (2) trivially implies (3). By Lemma 1.6, (3) implies (4).  $\square$

**Example 1.8.** Theorem 1.7 fails for Lindelöf linearly ordered topological spaces. Let  $X$  be  $(\omega_1 \times \mathbb{Z}) \cup (\{\omega_1\} \times \{0\})$  ordered lexicographically. Then  $X$  is Lindelöf and nonseparable and  $\{\{\langle \alpha, n \rangle\} : \alpha < \omega_1 \text{ and } n \in \mathbb{Z}\} \cup \{X \setminus (\alpha \times \mathbb{Z}) : \alpha < \omega_1\}$  is an  $\omega_1^{\text{op}}$ -like base of  $X$ .

## 2. A SPECTRUM OF NOETHERIAN TYPES

Theorem 1.7 implies that no linearly ordered compactum has Noetherian type  $\omega_1$ . What is the class of Noetherian types of linearly ordered compacta? We shall prove that an infinite cardinal  $\kappa$  is the Noetherian type of a linearly ordered compactum if and only if  $\kappa \neq \omega_1$  and  $\kappa$  is not weakly inaccessible.

**Theorem 2.1.** *Let  $\kappa$  be an uncountable cardinal and give  $\kappa + 1$  the order topology. If  $\kappa$  is regular, then  $Nt(\kappa + 1) = \kappa^+$ ; otherwise,  $Nt(\kappa + 1) = \kappa$ .*

*Proof.* Let  $\mathcal{A}$  be a base of  $\kappa + 1$  and let  $\lambda$  be a regular cardinal  $\leq \kappa$ . Let us show that  $\mathcal{A}$  is not  $\lambda^{\text{op}}$ -like. For every limit ordinal  $\alpha < \lambda$ , choose  $U_\alpha \in \mathcal{A}$  such that  $\alpha = \max U_\alpha$ ; choose  $\eta(\alpha) < \alpha$  such that  $[\eta(\alpha), \alpha] \subseteq U_\alpha$ . By Fodor's Theorem,  $\eta$  is constant on a stationary subset  $S$  of  $\lambda$ . Hence,  $\mathcal{A} \ni \{\eta(\min S) + 1\} \subseteq U_\alpha$  for all  $\alpha \in S$ ; hence,  $\mathcal{A}$  is not  $\lambda^{\text{op}}$ -like. Hence,  $Nt(\kappa + 1) \geq \kappa$  and  $Nt(\kappa + 1) > \text{cf } \kappa$ .

Moreover,  $Nt(\kappa + 1) \leq w(\kappa + 1)^+ = \kappa^+$ . Hence, it suffices to show that  $\kappa + 1$  has a  $\kappa^{\text{op}}$ -like base if  $\kappa$  is singular. Suppose  $E \in [\kappa]^{<\kappa}$  is unbounded in  $\kappa$ . Let  $F$  be the set of limit points of  $E$  in  $\kappa + 1$ . Define  $\mathcal{B}$  by

$$\mathcal{B} = \{(\beta, \alpha] : E \ni \beta < \alpha \in F \text{ or } \sup(E \cap \alpha) \leq \beta < \alpha \in \kappa \setminus F\}.$$

Then  $\mathcal{B}$  is a  $\kappa^{\text{op}}$ -like base of  $\kappa + 1$ .  $\square$

**Definition 2.2.** Given a poset  $P$  with ordering  $\leq$ , let  $P^{\text{op}}$  denote the set  $P$  with ordering  $\geq$ .

**Theorem 2.3.** *Suppose  $\kappa$  is a singular cardinal. Then there is a linearly ordered compactum with Noetherian type  $\kappa^+$ .*

*Proof.* Set  $\lambda = \text{cf } \kappa$  and  $X = \lambda^+ + 1$ . Partition the set of limit ordinals in  $\lambda^+$  into  $\lambda$ -many stationary sets  $\langle S_\alpha \rangle_{\alpha < \lambda}$ . Let  $\langle \kappa_\alpha \rangle_{\alpha < \lambda}$  be an increasing sequence of regular cardinals with supremum  $\kappa$ . For each  $\alpha < \lambda$  and  $\beta \in S_\alpha$ , set  $Y_\beta = (\kappa_\alpha + 1)^{\text{op}}$ . For each  $\alpha \in X \setminus \bigcup_{\beta < \lambda} S_\beta$ , set  $Y_\alpha = 1$ . Set  $Y = \bigcup_{\alpha \in X} \{\alpha\} \times Y_\alpha$  ordered lexicographically. Then  $Nt(Y) \leq w(Y)^+ \leq |Y|^+ = \kappa^+$ . Hence, it suffices to show that  $Y$  has no  $\kappa^{\text{op}}$ -like base.

Seeking a contradiction, suppose  $\mathcal{A}$  is a  $\kappa^{\text{op}}$ -like base of  $Y$ . For each  $\alpha < \lambda$ , let  $\mathcal{U}_\alpha$  be the set of all  $U \in \mathcal{A}$  that have at least  $\kappa_\alpha$ -many supersets in  $\mathcal{A}$ . Then, for all isolated points  $p$  of  $Y$ , there exists  $\alpha < \lambda$  such that  $\{p\} \notin \mathcal{U}_\alpha$ ; whence,  $p \notin \bigcup \mathcal{U}_\alpha$ . Since  $\langle \alpha + 1, 0 \rangle$  is isolated for all  $\alpha < \lambda^+$ , there exist  $\beta < \lambda$  and a set  $E$  of successor ordinals in  $\lambda^+$  such that  $|E| = \lambda^+$  and  $(E \times 1) \cap \bigcup \mathcal{U}_\beta = \emptyset$ . Let  $C$  be the closure of  $E$  in  $\lambda^+$ . Then  $C$  is closed unbounded; hence, there exists  $\gamma \in C \cap S_{\beta+1}$ . Set  $q = \langle \gamma, \kappa_{\beta+1} \rangle$ . Then  $q \in \overline{E \times 1}$ ; hence,  $q \notin \bigcup \mathcal{U}_\beta$ . Since  $q$  has coinitiality  $\kappa_{\beta+1}$ , any local base  $\mathcal{B}$  at  $q$  will contain an element  $U$  such that  $U$  has  $\kappa_\beta$ -many supersets in  $\mathcal{B}$ . Hence, there exists  $U \in \mathcal{U}_\beta$  such that  $q \in U$ ; hence,  $q \in \bigcup \mathcal{U}_\beta$ , which yields our desired contradiction.  $\square$

**Theorem 2.4.** *No linearly ordered compactum has weakly inaccessible Noetherian type.*

*Proof.* Suppose  $\kappa$  is weakly inaccessible and  $X$  is a linearly ordered compactum satisfying  $Nt(X) \leq \kappa$ . Then it suffices to prove  $Nt(X) < \kappa$ . By Lemma 1.6, we have  $\pi(X) = d(X) < \kappa$ . If  $w(X) \geq \kappa$ , then  $Nt(X) > \kappa$  by Proposition 1.4, in contradiction with our assumptions about  $X$ . Hence,  $w(X) < \kappa$ ; hence,  $Nt(X) \leq w(X)^+ < \kappa$ .  $\square$

### 3. SUBSETS OF BASES

Given a space  $X$ , does every base of  $X$  contain an  $Nt(X)^{\text{op}}$ -like base of  $X$ ? There is no known counterexample, and Lemma 3.2 below says the answer is yes for the wide class of spaces  $X$  satisfying  $\chi(p, X) = w(X)$  for all  $p \in X$ . We will present some further partial answers to this question. In particular, it is consistent that the answer is yes for all homogeneous compacta.

**Proposition 3.1.** *If  $X$  is a space and  $\mathcal{A}$  is  $(w(X)^+)^{\text{op}}$ -like base of  $X$ , then  $|\mathcal{A}| \leq w(X)$ .*

*Proof.* Seeking a contradiction, suppose  $|\mathcal{A}| > w(X)$ . Let  $\mathcal{B}$  be a base of  $X$  of size  $w(X)$ . Then every element of  $\mathcal{A}$  contains an element of  $\mathcal{B}$ . Hence, some  $U \in \mathcal{B}$  is contained in  $w(X)^+$ -many elements of  $\mathcal{A}$ . Clearly  $U$  contains some  $V \in \mathcal{A}$ , so  $\mathcal{A}$  is not  $(w(X)^+)^{\text{op}}$ -like.  $\square$

**Lemma 3.2.** *Suppose  $X$  is a space with no isolated points and  $\chi(p, X) = w(X)$  for all  $p \in X$ . Further suppose  $\kappa = \text{cf } \kappa \leq \min\{Nt(X), w(X)\}$  and  $X$  has a network consisting of at most  $w(X)$ -many  $\kappa$ -compact sets. Then every base of  $X$  contains an  $Nt(X)^{\text{op}}$ -like base of  $X$ .*

*Proof.* Set  $\lambda = Nt(X)$  and  $\mu = w(X)$ . Let  $\mathcal{A}$  be an arbitrary base of  $X$ ; let  $\mathcal{B}$  be a  $\lambda^{\text{op}}$ -like base of  $X$ ; let  $\mathcal{N}$  be a network of  $X$  consisting of at most  $\mu$ -many  $\kappa$ -compact sets. By Proposition 3.1,  $|\mathcal{B}| \leq \mu$ . Let  $\langle \langle N_\alpha, B_\alpha \rangle \rangle_{\alpha < \mu}$  enumerate  $\{\langle N, B \rangle \in \mathcal{N} \times \mathcal{B} : N \subseteq B\}$ . Construct a sequence  $\langle \mathcal{G}_\alpha \rangle_{\alpha < \mu}$  as follows. Suppose  $\alpha < \mu$  and  $\langle \mathcal{G}_\beta \rangle_{\beta < \alpha}$  is a sequence of elements of  $[\mathcal{B}]^{< \kappa}$ . For each  $p \in N_\alpha$ , we have  $\chi(p, X) = \mu \geq \kappa = \text{cf } \kappa$ ; hence, we may choose  $U_{\alpha, p} \in \mathcal{B}$  such that  $p \in U_{\alpha, p} \notin \bigcup_{\beta < \alpha} \mathcal{G}_\beta$ . Choose  $\sigma_\alpha \in [N_\alpha]^{< \kappa}$  such that  $N_\alpha \subseteq \bigcup_{p \in \sigma_\alpha} U_{\alpha, p}$ . Set  $\mathcal{G}_\alpha = \{U_{\alpha, p} : p \in \sigma_\alpha\}$ .

For each  $\alpha < \mu$ , choose  $\mathcal{F}_\alpha \in [\mathcal{A}]^{< \kappa}$  such that  $N_\alpha \subseteq \bigcup \mathcal{F}_\alpha \subseteq B_\alpha$  and  $\mathcal{F}_\alpha$  refines  $\mathcal{G}_\alpha$ . Set  $\mathcal{F} = \bigcup_{\alpha < \mu} \mathcal{F}_\alpha$ , which is easily seen to be a base of  $X$ . Let us show that  $\mathcal{F}$  is  $\lambda^{\text{op}}$ -like. Suppose not. Then, since  $\kappa = \text{cf } \kappa \leq \lambda$ , there exist  $V \in \mathcal{F}$  and  $I \in [\mu]^\lambda$  and  $\langle W_\alpha \rangle_{\alpha \in I} \in \prod_{\alpha \in I} \mathcal{F}_\alpha$  such that  $V \subseteq \bigcap_{\alpha \in I} W_\alpha$ . For each  $\alpha \in I$ , there is a superset of  $W_\alpha$  in  $\mathcal{G}_\alpha$ . By induction,  $\mathcal{G}_\alpha \cap \mathcal{G}_\beta = \emptyset$  for all  $\alpha < \beta < \mu$ ; hence,  $V$  has  $\lambda$ -many supersets in the  $\lambda^{\text{op}}$ -like base  $\mathcal{B}$ , which is absurd, for  $V$  has a subset in  $\mathcal{B}$ .  $\square$

**Theorem 3.3.** *If  $X$  is a compactum and  $\pi\chi(p, X) < \text{cf } \kappa = \kappa \leq w(X)$  for all  $p \in X$ , then  $Nt(X) > \kappa$ .*

*Proof.* Let  $\mathcal{A}$  be a base of  $X$ . By Misčenko's Lemma, there exist  $p \in X$  and  $\mathcal{B} \in [\mathcal{A}]^\kappa$  such that  $p \in \bigcap \mathcal{B}$ . Let  $\mathcal{C} \in [\mathcal{A}]^{< \kappa}$  be a local  $\pi$ -base at  $p$ . Then some element of  $\mathcal{C}$  is contained in  $\kappa$ -many elements of  $\mathcal{B}$ . Hence,  $\mathcal{A}$  is not  $\kappa^{\text{op}}$ -like.  $\square$

**Corollary 3.4.** *If  $X$  is a homogeneous compactum with regular weight, then every base of  $X$  contains an  $Nt(X)^{\text{op}}$ -like base.*

*Proof.* If  $\chi(X) = w(X)$ , then just apply Lemma 3.2. If  $\chi(X) < w(X)$ , then  $Nt(X) = w(X)^+$  by Lemma 3.3; whence, every base  $\mathcal{A}$  of  $X$  contains a base  $\mathcal{B}$  that is  $Nt(X)^{\text{op}}$ -like simply because  $|\mathcal{B}| < Nt(X)$ .  $\square$

We can exchange the above requirement that  $w(X)$  be regular for a weak form of GCH.

**Corollary 3.5.** *Suppose every limit cardinal is strong limit. Then, for every homogeneous compactum  $X$ , every base of  $X$  contains an  $Nt(X)^{\text{op}}$ -like base.*

*Proof.* By Archangelskii's Theorem,  $\chi(X) \leq w(X) \leq 2^{\chi(X)}$ . If  $\chi(X) < w(X)$ , then  $w(X)$  is a successor cardinal; apply Theorem 3.4. If  $\chi(X) = w(X)$ , then apply Lemma 3.2.  $\square$

If we don't assume homogeneity, then we still have the following weak result.

**Theorem 3.6.** *Suppose  $\kappa$  is a regular cardinal and  $X$  is a locally  $\kappa$ -compact  $T_3$  space such that  $Nt(X) \leq w(X) = \kappa$ . Then every base of  $X$  contains a  $\kappa^{\text{op}}$ -like base of  $X$ .*

*Proof.* Let  $\mathcal{A}$  be a base of  $X$  and let  $\mathcal{B}$  be a  $\kappa^{\text{op}}$ -like base of  $X$ . We may assume  $|\mathcal{A}| = |\mathcal{B}| = \kappa$ . Suppose  $\kappa = \omega$ . Then  $X$  is a metrizable; fix a compatible metric. Moreover, there is a sequence  $\langle U_n \rangle_{n < \omega}$  of open subsets of  $X$  such that  $X = \bigcup_{n < \omega} U_n$  and  $\bar{U}_n$  is compact for all  $n < \omega$ . For each  $\langle m, n \rangle \in \omega^2$ , let  $\mathcal{A}_{m, n}$  be a finite cover

of  $\bar{U}_m$  by elements of  $\mathcal{A}$  with diameter less than  $2^{-n}$ . Set  $\mathcal{A}' = \bigcup_{m < n < \omega} \mathcal{A}_{m,n}$ . Then  $\mathcal{A}' \subseteq \mathcal{A}$  and  $\mathcal{A}'$  is a base of  $X$ . Suppose  $V, W \in \mathcal{A}'$  and  $V \not\subseteq W$ . Then  $V$  contains a ball of radius  $2^{-n}$  for some  $n < \omega$ ; hence,  $\text{diam } W \geq 2^{-n}$ ; hence,  $W \in \bigcup_{l < m < n} \mathcal{A}_{l,m}$ . Thus,  $\mathcal{A}'$  is  $\omega^{\text{op}}$ -like.

Suppose  $\kappa > \omega$ . Let  $\langle M_\alpha \rangle_{\alpha < \kappa}$  be a continuous elementary chain such that  $\{M_\beta : \beta < \alpha\} \cup \{\mathcal{A}, \mathcal{B}\} \subseteq M_\alpha \prec H_\theta$  and  $|M_\alpha| < \kappa$  and  $M_\alpha \cap \kappa \in \kappa$  for all  $\alpha < \kappa$ . Then  $\mathcal{A} \cup \mathcal{B} \subseteq M_\kappa$ . For each  $\alpha < \kappa$ , let  $\mathcal{U}_\alpha$  denote the set of all  $U \in \mathcal{A} \cap M_{\alpha+1}$  for which  $U$  has a superset in  $\mathcal{B} \setminus M_\alpha$ . Set  $\mathcal{U} = \bigcup_{\alpha < \kappa} \mathcal{U}_\alpha \subseteq \mathcal{A}$ . First, let us show that  $\mathcal{U}$  is  $\kappa^{\text{op}}$ -like. Suppose  $\alpha < \kappa$  and  $\mathcal{U}_\alpha \ni U \subseteq V \in \mathcal{U}$ . Then there exist  $\beta < \kappa$  and  $B \in \mathcal{B} \setminus M_\beta$  such that  $B \supseteq V \in M_{\beta+1}$ . Hence,  $U \subseteq B$ ; hence,  $B \in \{W \in \mathcal{B} : U \subseteq W\} \in M_{\alpha+1} \cap [\mathcal{B}]^{< \kappa}$ ; hence,  $B \in M_{\alpha+1}$ ; hence,  $\beta \leq \alpha$ ; hence,  $V \in M_{\alpha+1}$ . Thus,  $\mathcal{U}$  is  $\kappa^{\text{op}}$ -like.

Finally, let us show that  $\mathcal{U}$  is a base of  $X$ . Suppose  $p \in B \in \mathcal{B}$  and  $\bar{B}$  is  $\kappa$ -compact. Then it suffices to find  $U \in \mathcal{U}$  such that  $p \in U \subseteq B$ . Let  $\beta$  be the least  $\alpha < \kappa$  such that there exists  $A \in \mathcal{A} \cap M_{\alpha+1}$  satisfying  $p \in A \subseteq \bar{A} \subseteq B$ . Fix such an  $A$ . If  $B \notin M_\beta$ , then  $A \in \mathcal{U}_\beta$  and  $p \in A \subseteq B$ . Hence, we may assume  $B \in M_\beta$ . For each  $q \in \bar{A}$ , choose  $\langle A_q, B_q \rangle \in \mathcal{A} \times \mathcal{B}$  such that  $q \in A_q \subseteq B_q \subseteq \bar{B}_q \subseteq B$ . Then there exists  $\sigma \in [\bar{A}]^{< \kappa}$  such that  $\bar{A} \subseteq \bigcup_{q \in \sigma} A_q$ . By elementarity, we may assume  $\langle \langle A_q, B_q \rangle \rangle_{q \in \sigma} \in M_{\beta+1}$ ; hence,  $A_q, B_q \in M_{\beta+1}$  for all  $q \in \sigma$ . Choose  $q \in \sigma$  such that  $p \in A_q$ . If  $B_q \notin M_\beta$ , then  $A_q \in \mathcal{U}_\beta$  and  $p \in A_q \subseteq B$ . Hence, we may assume  $B_q \in M_\beta$ ; hence, we may choose  $\alpha < \beta$  such that  $B_q \in M_{\alpha+1}$ . Then  $B \in M_{\alpha+1}$ , for  $\mathcal{B}$  is  $\kappa^{\text{op}}$ -like. For each  $r \in \bar{B}_q$ , choose  $W_r \in \mathcal{A}$  such that  $r \in W_r \subseteq \bar{W}_r \subseteq B$ . Then there exists  $\tau \in [\bar{B}_q]^{< \kappa}$  such that  $\bar{B}_q \subseteq \bigcup_{r \in \tau} W_r$ . By elementarity, we may assume  $\langle W_r \rangle_{r \in \tau} \in M_{\alpha+1}$ . Choose  $r \in \tau$  such that  $p \in W_r$ . Then  $W_r \in \mathcal{A} \cap M_{\alpha+1}$  and  $p \in W_r \subseteq \bar{W}_r \subseteq B$ , in contradiction with the minimality of  $\beta$ . Thus,  $\mathcal{U}$  is a base of  $X$ .  $\square$