RESEARCH STATEMENT

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1. Introduction

Of the different subfields of mathematics, I am most fascinated by set theory, especially set-theoretic topology, which is the subject of my doctoral dissertation. I plan to continue my research is this subfield as well broaden my work, making contributions to other ares of set theory, as well as finite combinatorics. I am also very open to the idea of advancing other areas of mathematical knowledge.

For the reader seeking a gentler, more informal introduction to set-theoretic topology and my work therein, I have appended a slide presentation to this research statement.

2. Motivating questions

My highest-priority research goal has long been van Douwen's Problem about large homogeneous compacta. (Compactum is short for compact Hausdorff space). A compactum X is *homogeneous* if and only if for every two points there is a homeomorphism from X to X sending one point to the other. Now, what does "large" mean in this context? Every power 2^κ of the discrete two-point space is a homegeneous compactum, and we can choose one with as large a cardinality as we wish. Less trivially, all infinite powers $[0,1]^\kappa$ of the unit interval are homogeneous compacta [3], and every compactum can be embedded as a closed subspace of such a power. However, here "large" does not refer to cardinality or embeddability, but to cellularity. The *cellularity* $c(X)$ of a space X is defined as the supremum of the cardinalities of its pairwise disjoint families of open subsets of X. The spaces 2^κ and $[0,1]^\kappa$ are actually small by this measure: all their pairwise disjoint families of open sets are countable, so their cellularity is only $\aleph_0$. It was once unkown whether there was a homogeneous compactum with uncountable cellularity, but thanks to a 1964 result of Maurice [8], we now have a simple example of such a space. If the binary sequences of length $\omega\cdot\omega$ are ordered lexicographically (and given the order topology), then the result a homogeneous compactum that has a family of $2^{\aleph_0}$-many pairwise disjoint open sets. Van Douwen's Problem [5], which is still open in all models of the standard ZFC axioms of set theory, asks, "is there a homogeneous compactum with cellularity exceeding $2^{\aleph_0}$?"

Definition 1. A *local base* at a point p in a space X is a family of neighborhoods of p such that every neighborhood U of p contains a neighborhood in the local base. The *character* $\chi(p,X)$ of p is the least κ such that p has a local base of size κ. The character $\chi(X)$ of X is $\sup_{p\in X}\chi(p,X)$. We say X is *first countable* if $\chi(X)\leq\aleph_0$, *i.e.*, if every point has a countable local base.

Van Douwen's Problem is hard because there are very few ways to construct homogeneous compacta that even have more than $2^{\aleph_0}$-many points. There is a zoo of constructions of homogeneous compacta which are first countable. However, all first countable compacta have at most $2^{\aleph_0}$-many points by Arhangel′skiĭ's Theorem [1]. Moreover, $|X|\leq 2^{\chi(X)}$ for all infinite compacta X.

The other main class of known homogeneous compacta is the class of compact groups. These groups can have an arbitrarily large number of points, but they cannot have an uncountable family of pairwise disjoint open sets. To see this, note that each nonempty open subset of a compact group has positive Haar measure, and that the whole group has finite Haar measure. More generally, every compact groups is *dyadic* [6] (that is, a continuous image of a power of 2), and it is known that an uncountable family of open subsets of a dyadic compactum must strongly fail to be pairwise disjoint: such a family has an uncountable subfamily with nonempty intersection.

Definition 2. A *base* of a space X is a family of open sets that includes a local base at every point. The *weight* $w(X)$ of X is the least κ such that X has a base of size κ.

Taking products of arbitrarily many first-countable compacta and dyadic compacta yields no spaces with cellularity exceeding $2^{\aleph_0}$, and it exhuasts almost all known examples of homogeneous compacta. Only two other, relatively narrow classes of homogeneous compacta are known to exist. They were discovered respectively by Jan van Mill [17] and myself [9] in last few years. (Actually, in some models of ZFC, van Mill's exception does not exist, leaving only my exception.) These compacta also fail to have cellularity exceeding $2^{\aleph_0}$. In fact, all known homogeneous compacta are continuous images of products of compacta each with weight at most $2^{\aleph_0}$, and it is easy to show (using calibers [13]) that such images can never have cellularity exceeding $2^{\aleph_0}$. To solve van Douwen's Problem positively, one would have to find an "exotic" homogeneous compacta that is not such an image.

My progress to date on van Douwen's Problem, besides the discovery of an exceptional class of homogeneous compacta, has been indirect, investigating some order-theoretic invariants of topological spaces.

Definition 3. Given a cardinal κ, we say a family $\mathcal{F}$ of open sets is κ^{op}-like if every $U \in \mathcal{F}$ has fewer than κ-many supersets in $\mathcal{F}$. Let the *Noetherian type* $Nt(X)$ denote the least κ such that X has κ^{op}-like base. Given $p \in X$, let the *local Noetherian type* $\chi Nt(p, X)$ of p denote the least κ such that p has a κ^{op}-like local base. The local Noetherian type $\chi Nt(X)$ of X is $\sup_{p\in X} \chi Nt(p, X)$.

Consider a simple example. Every compact metric space X satisfies $Nt(X) \leq \aleph_0$ because if for each $n \in \mathbb{N}$ we choose a finite cover $\mathcal{F}_n$ of X by open balls of radius 2^{-n}, then $\bigcup_{n<\omega} \mathcal{F}_n$ is a base of X and every ball $B \in \bigcup_{n<\omega} \mathcal{F}_n$ has only finitely many supersets in $\bigcup_{n<\omega} \mathcal{F}_n$. (Why? If B has radius 2^{-n}, then every proper superset of B has diameter at least 2^{-n}, and is therefore not in $\bigcup_{m>n} \mathcal{F}_m$.)

Next are some highlights from my results in [10] about Noetherian type.

Theorem 4. *All known homogeneous compacta satisfy* $\chi Nt(X) \leq \aleph_0$. *Moreover,* GCH *implies that* $\chi Nt(X) \leq c(X)$ *for all homogeneous compacta* X *(known and unknown).*

I also constructed a nonhomogeneous compactum satisfying $\chi Nt(X) > c(X) = \aleph_0$ (without assuming GCH).

Theorem 5. *Suppose* κ *is an uncountable regular cardinal and* X *is a homogeneous compactum and a continuous image of a product* $\prod_{i\in I} Y_i$ *of compacta such that* $w(Y_i) < \kappa$ *for all* $i \in I$. *Then* $Nt(X) \leq \kappa$. *In particular, every known homogeneous compactum* X *satisfies* $Nt(X) \leq \left(2^{\aleph_0}\right)^+$.

Additionally, I showed that if X is the double-arrow space ($[0, 1]$ with each interior point expanded to two points), then $Nt(X) = \left(2^{\aleph_0}\right)^+$ (so every base has an element with $2^{\aleph_0}$-many supersets in the base). It is easy to show that the double-arrow space is a homogeneous compactum. It is also easy to build nonhomogeneous spaces with arbitarily large Noetherian type: given an infinite cardinal κ, if we choose a point from 2^κ and identify it with a point from 2^{κ^+}, then the resulting quotient of $2^\kappa \oplus 2^{\kappa^+}$ has Noetherian type κ^{++}.

If X is a dyadic compactum, homogeneous or not, then $\chi Nt(X) = \aleph_0$. If X is a homogeneous dyadic compactum (for example, a compact group), then $Nt(X) = \aleph_0$.

Definition 6. A *local* π*-base* at a point p in a space X is a family of nonempty open sets such that every neighborhood U of p contains a member of the local π-base. The π*-character* $\pi\chi(p, X)$ of p is the least κ such that p has a local π-base of size κ. The π-character $\pi\chi(X)$ of X is $\sup_{p\in X} \pi\chi(p, X)$.

Definition 7. A *π-base* of a space X is a family of nonempty open sets that includes a local π-base at every point. The π-weight $\pi(X)$ of X is the least κ such that X has a π-base of size κ. The Noetherian π-type $\pi Nt(X)$ of X is the least κ such that X has a κ^{op}-like π-base.

I also showed that every known homogeneous compactum X satisfies $\pi Nt(X) \leq \aleph_1$. Thus, van Douwen's Problem now has three cousins:

Question 8. Is there a homogeneous compactum X satisfying $c(X) > 2^{\aleph_0}$? $Nt(X) > \left(2^{\aleph_0}\right)^+$? $\chi Nt(X) > \aleph_0$? $\pi Nt(X) > \aleph_1$?

In proving Theorem 5, the use of elementary substructures, a powerful technique of logic, was the crucial tool. Interestingly, it did not suffice to work with a single elementary chain. The desired κ^{op}-like base for Theorem 5 was constructed one countable piece at a time, with each piece being a certain countable base of the metrizable quotient space naturally induced by a countable elementary substructure. More specifically, I used a sequence $\langle M_\alpha\rangle_{\alpha<\eta}$ of countable elementary substructures of H_θ, the set of all sets with transitive closure of size less than θ, where θ is a sufficiently large regular cardinal. Requiring only that $\langle M_\beta\rangle_{\beta<\alpha} \in M_\alpha$ for all α, I generalized a technique of Jackson and Mauldin [4] to construct a sequence $\langle \Sigma_\alpha\rangle_{\alpha\leq\eta}$ of finite sets of (possibly uncountable) elementary substructures of H_θ such that $\bigcup \Sigma_\alpha = \bigcup_{\beta<\alpha} M_\beta$ and $\Sigma_\alpha \subseteq M_\alpha$ for all α.

I believe that if van Douwen's Problem has a negative solution, then elementary substructures will be heavily used in the first proof of this. Order-theoretic properties of local bases, such as local Noetherian type, are also promising candidate techniques for a negative solution to the problem because they provide a new way to show that a space has two points sufficiently different that the space cannot be homogeneous.

In fact, Tukey classes can be used to make even finer distinctions than those made by local Noetherian type. Given any two directed sets P and Q, we say that P is *Tukey reducible* to Q and write $P \leq_T Q$ if there exists a map $f\colon P \to Q$ such that every set bounded above in Q has f-preimage bounded above in P. We say P and Q are *Tukey equivalent* and write $P \equiv_T Q$ if $P \leq_T Q \leq_T P$. Tukey showed that $P \equiv_T Q$ if and only if P and Q embed as cofinal subsets of a common directed set [16]. Getting back to topology, it is easy to see that $h\colon X \to Y$ is a homeomorphism and $p \in X$, then $\langle \mathcal{A}, \supseteq\rangle \equiv_T \langle \mathcal{B}, \supseteq\rangle$ for every local base $\mathcal{A}$ at p and every local base $\mathcal{B}$ at $h(p)$. Moreover, we have $\chi Nt(p, X) \leq \kappa$ if and only if every local base $\mathcal{A}$ at p satisfies $\langle \mathcal{A}, \supseteq\rangle \geq_T \langle [\chi(p, X)]^{<\kappa}, \subseteq\rangle$. In particular, every known homogeneous compactum X only has local bases Tukey equivalent to $[\chi(X)]^{<\omega}$.

It is known that if X is a compactum such that every point has a local base that is linearly ordered (equivalently, well ordered) by $\supseteq$, then some point in X has countable character. I proved the following analog for π-character by combining Tukey reducibility with some of my results about local Noetherian type.

Theorem 9. *Suppose X is a compactum such that every point p has a local base $\mathcal{B}_p$ that lacks an uncountable subset of pairwise $\supseteq$-incomparable elements. Then some point in X has countable π-character.*

In the last few decades, some of the strongest independence results of set-theoretic topology have used the technique of proper forcing to show that there are models of ZFC in which certain topological objects of size $\aleph_1$ do not exist. Indeed, the Proper Forcing Axiom (PFA) implies that for a point with character $\aleph_1$, every local base at that point, when ordered by $\supseteq$, is Tukey equivalent to one of $\langle [\omega_1]^{<\omega}, \subseteq\rangle$, $\langle \omega \times \omega_1, \leq \times \leq\rangle$, and $\langle \omega_1, \leq\rangle$ (Todorčević [15]). Moreover, in a homogeneous compactum, no local base is Tukey equivalent to ω_1 (because every point is a limit point of a countably infinite set). Thus, there are only two possible Tukey classes. However, only one of these two, $[\omega_1]^{<\omega}$, is known to occur in a homogeneous compactum. I would love to answer whether PFA rules out the other Tukey class.

Question 10. Does PFA, or even ZFC alone, refute the existence of a homogeneous compactum in which some (equivalently, every) local base $\mathcal{B}$ satisfies $\langle \mathcal{B}, \supseteq \rangle \equiv_T \omega \times \omega_1$?

An affirmative answer to this question would be an important first step towards proving that $\chi Nt(X) = \aleph_0$ for every homogeneous compactum X. A counterexample would be even more interesting, especially if it suggested a way to solve van Douwen's Problem positively.

I have also investigated Tukey classes of local bases in $\beta\mathbb{N}\setminus\mathbb{N}$ [12]. It is known that the Continuum Hypothesis (CH) implies $\beta\mathbb{N} \setminus \mathbb{N}$ has a local base Tukey equivalent (and order isomorphic) to ω_1. It is also known that ZFC alone implies $\beta\mathbb{N} \setminus \mathbb{N}$ has a local base Tukey equivalent to $\langle [2^{\aleph_0}]^{<\omega}, \subseteq \rangle$. Assuming $\Diamond$, which is stronger than CH, I proved that there is a local base in $\beta\mathbb{N} \setminus \mathbb{N}$ not Tukey equivalent to either of ω_1 and $[2^{\aleph_0}]^{<\omega}$. This result prompts some interesting questions that I would like to answer in the future. Also, insights about ultrafilters on ω may lead to insights about homogeneous compacta. The strongest precedent for this claim is Kunen's proof that products of compact F-spaces are not homogeneous [5]. The proof uses the existence of two weak P-points in $\beta\mathbb{N} \setminus \mathbb{N}$ that are Rudin-Keisler incomparable.

Question 11. Does CH already imply that there are three pairwise Tukey inequivalent local bases in $\beta\mathbb{N} \setminus \mathbb{N}$?

If the answer to the latter question is "no," then this can probably be proved using a proper forcing extension.

Saharon Shelah [14] has shown that there is a proper forcing extension in which $\beta\mathbb{N} \setminus \mathbb{N}$ has no P-points, that is, points with local bases that are σ-directed with respect to $\supseteq$. In particular, no local base is Tukey equivalent to ω_1. It is natural to ask whether this result can be strengthened.

Question 12. Is there a model of ZFC in which $\chi Nt(\beta\mathbb{N} \setminus \mathbb{N}) = \aleph_0$? Is there one in which every local base in $\beta\mathbb{N} \setminus \mathbb{N}$ is Tukey equivalent to $[2^{\aleph_0}]^{<\omega}$?

My research about Noetherian type and its cousins naturally led to many questions about $Nt(\beta\omega\setminus\omega)$ and $\pi Nt(\beta\omega \setminus \omega)$, as well as $\chi Nt(\beta\omega \setminus \omega)$. In [11], I obtained many independence results about these cardinals. For example, $Nt(\beta\omega \setminus \omega)$ is at least $\mathfrak{s}$, but can consistently be $\aleph_1$, $2^{\aleph_0}$, $\left(2^{\aleph_0}\right)^+$, or strictly between $\aleph_1$ and $2^{\aleph_0}$. $Nt(\beta\omega \setminus \omega)$ is closely related to the existence of special kinds of splitting families. (A splitting family is a set S such that for all infinite $x \subseteq \omega$, there is an $y \in S$ for which $x \cap y$ and $x \setminus y$ are infinite.)

3. Goals

With respect to set-theoretic topology, my primary research goal is to solve van Douwen's Problem. My secondary research goal is to characterize the spectrum of Tukey classes of local bases that occur in homogeneous compacta. The latter goal both appears easier to attain than the former and likely to yield partial results about van Douwen's Problem, if not its outright solution.

Along the way, I expect to continue to contribute to combinatorial set theory. Infinitary combinatorics and set-theoretic topology has always been intimately intertwined. I am also fascinated by the inner model program in set theory.

Additionally, I am interested in finding new applications of set theory to other areas of mathematics. For example, there is a particularly striking application of set theory (specifically, n-subtle cardinals) to finite combinatorics due to Friedman [2]. Instead of reproducing the precise theorem and its prerequisite definitions, I will quote Friedman's proof strategy.

> We start with a discrete (or finite) structure X obeying certain hypotheses H. We wish to prove that a certain kind of finite configuration occurs in X, assuming that H holds. We define a suitable concept of completion in the context of arbitrary linearly ordered sets. We verify that if X has a completion with the desired kind

of finite configuration, then X already has the desired kind of finite configuration. We then show, using hypotheses H, that X has completions of every well-ordered type. We now use the existence of a suitably large cardinal λ. Using large cardinal combinatorics, we show that in any completion of order type λ, the desired kind of finite configuration exists. Hence the desired kind of finite configuration already exists in X.

I am interested both in applying heuristics like the one above and in finding new heuristics for connecting the transfinite to the finite.

I also plan to attack some questions in finite combinatorics that do not appear to need any infinitary set theory to solve them. I am particularly interested in problems of Ramsey theory and combinatorial game theory. For example, as far as I can tell, the following two questions are open and interesting.

Question 13. According to Li's definition of three-player impartial games [7], if G is impartial, does the last player always have a winning strategy for $G+G+G$? (The well-known answer to the analogous question for two players is "yes.")

Question 14. Does every finite K_4-free graph have a binary coloring of edges with no monochromatic triangles?

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